I. Transmission lines and (introduction) antennas

1. Transmission lines, sinusoidal waveforms

Transmission lines are used to transmit electric energy and signals from one point to another. The basic transmission line connects a source to a load. This may be a transmitter and a antenna, a shift register and the memory in a digital computer, a hydro-electric generating plant and a substation several hundred miles away, a television antenna and a receiver, or one channel of a stereo turntable and one input of the preamplifier, ...
In classic networktheory we assume that the physical dimensions of a network are small enough with regard to the wavelength λ, so no standing wave patterns do appear. This isn't the case with long lines or high frequencies! Between start- and endpoint of the line there is a distance amount to some (or a part of a) wavelength(s).
The line is now a network with *distributed parameters* and has a *characteristic impedance* Z_c.
The important fact is that forward and backward waves can be sustained. The presence of backward as well as forward waves evidently affects such things as transmitted power, distribution of electric field, and losses in a transmission line. In transmission-line theory, techniques are developed for conviently studying such effects, and the results are applicable to any form of one-dimensional wave motion.
We start by investigating the appropiate description of a uniform line in circuit terms.

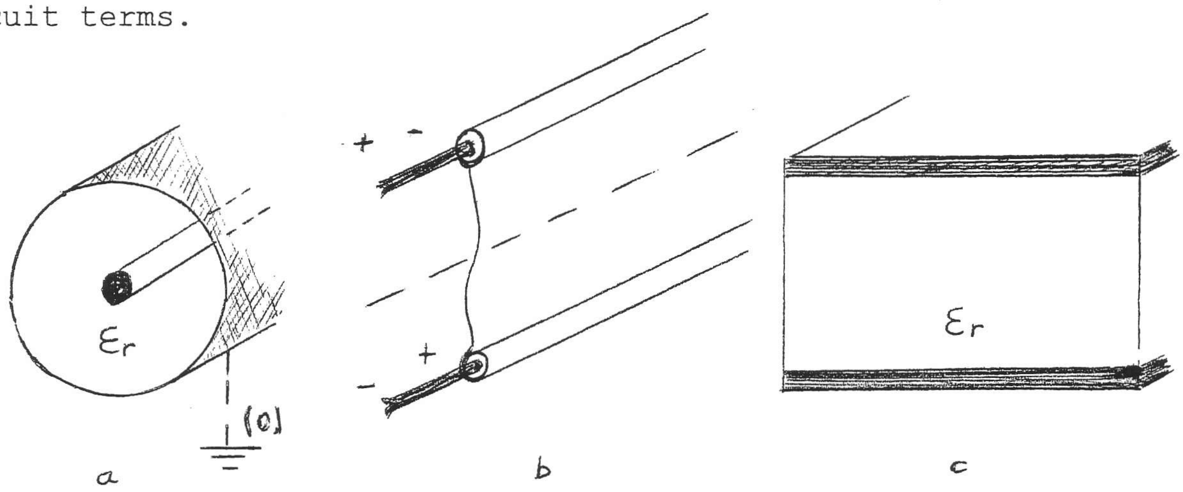


Fig.1-1: The geometry of the (a) coaxial, (b) two-wire, and (c) planar transmission lines. Homogeneous dielectrics are assumed.

1.1 Voltage and current equations

Analogous with the propagation of a plane electromagnetic wave in a homogeneous medium we can consider a length of uniform two-conductor line connected between a source and a load. Within any short length we can discern both energy storage and energy dissipation: the latter occurs both in the conductors and in the dielectric, and the former as both magnetic and electrostatic energy. A circuit displaying these properties is shown in *fig.1-2*. The parameters $\mathcal{L},\mathbb{C},r,g$ will be measured per unit length $\boldsymbol{\delta x}$.

The stored energies are $\frac{1}{2}\mathcal{L}i^2.\delta x$, $\frac{1}{2}\mathbb{C}v^2.\delta x$, and the energy dissipation $(ri^2 + gv^2).\delta z$, which are compatible with ideas of energy distributed along the line.

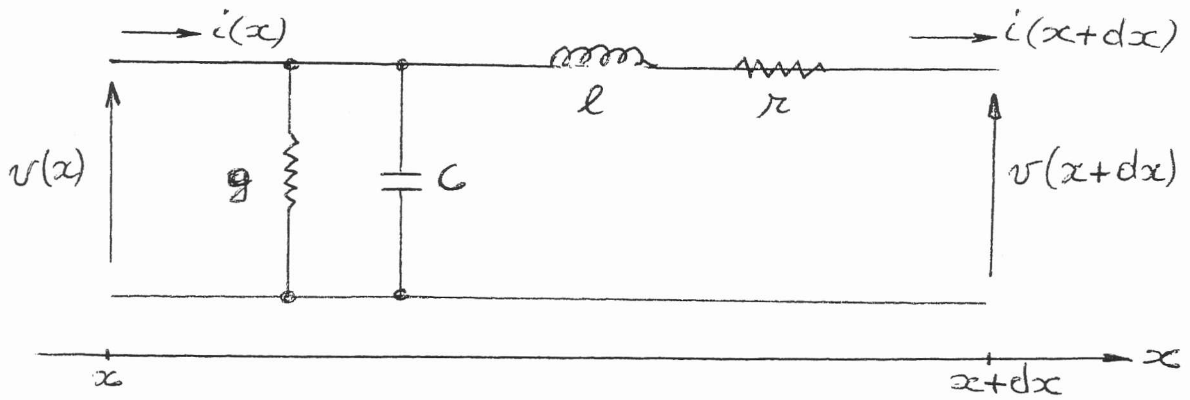


Fig.1-2: Circuit equivalent to elementary length of transmission line.

With Ohm's law, and series-impedance $z=r+j\omega\mathcal{L}$
shunt-admittance $y=g+j\omega\mathbb{C}$, we have

$$v(x+\delta x)-v(x) = -z.i(x)\delta x \quad (1.1)$$

$$i(x)-i(x+\delta x) = +y.v(x)\delta x \quad (1.2)$$

Now we get two partial differential-equations:

$$\frac{\delta v}{\delta x} = -z.i \quad (1.3) \qquad \frac{\delta i}{\delta x} = -y.v \quad (1.4)$$

Deriving with respect to x gives:

$$\frac{\delta^2 v}{\delta x^2} = -z\frac{\delta i}{\delta x} = +zy.v = \gamma^2.v \quad (1.5)$$

$$\frac{\delta^2 i}{\delta x^2} = -y\frac{\delta v}{\delta x} = +zy.i = \gamma^2.i \quad (1.6)$$

We can write the solution of those 2nd order diff.eq. as:

$$v(x) = A_1 e^{\gamma x} + B_1 e^{-\gamma x} \qquad (1.7)$$

$$i(x) = A_2 e^{\gamma x} + B_2 e^{-\gamma x} \qquad (1.8)$$

With propagation constant

$$\gamma = +\sqrt{zy} = \alpha + j\beta \qquad (1.9)$$

By substitution in (1.7) or (1.8) we get the solutions

$$\{e^{\alpha x},\ e^{j\beta x},\ e^{-\alpha x},\ e^{-j\beta x}\} \qquad (1.10)$$

Interpretation of those terms:

- $e^{\alpha x}$ = magnitudefactor, which indicates that the magnitude takes off from higher to lower x values (so in -x direction).
- $e^{j\beta x}$ = or a sinusoidal waveform with respect to {x,t}: $e^{j(\beta x+\omega t)}$, representing a waveform travelling in -x direction with speed ω/β.
- $e^{-\alpha x}$ = conform to $e^{+\alpha x}$, but +x direction.
- $e^{-j\beta x}$ = conform to $e^{+j\beta x}$, but +x direction.

Now we have to determine constants A_1, A_2, B_1 and B_2.
Substituting (1.3) in (1.7) gives

$$\frac{\delta v}{\delta x} = \gamma A_1 e^{\gamma x} + (-\gamma) B_1 e^{-\gamma x} = -z.i = -z.A_2 e^{\gamma x} - z.B_2 e^{-\gamma x} \qquad (1.11)$$

Equalization of the exponents

$$A_2 = -\sqrt{\frac{y}{z}} A_1 \qquad (1.12) \qquad\qquad B_2 = +\sqrt{\frac{y}{z}} B_1 \qquad (1.13)$$

So we can write the solution as

$$v(x) = A_1 e^{\gamma x} + B_1 e^{-\gamma x} \qquad (1.14)$$

$$i(x) = \sqrt{\frac{y}{z}} (-A_1 e^{\gamma x} + B_1 e^{-\gamma x}) \qquad (1.15)$$

Considering an **infinite** transmission line with a source at x=0, there will only be a voltage (current) waveform in +x direction.
Now the eq. (1.14) and (1.15) convert to

$$v(x) = B_1 e^{-\gamma x} \quad (1.16) \qquad i(x) = \sqrt{\frac{y}{z}} B_1 e^{-\gamma x} \quad (1.17)$$

Ratio of (1.16) and (1.17) gives

$$z(x) = \frac{v(x)}{i(x)} = \sqrt{\frac{z}{y}} \quad (1.18)$$

The **input** impedance of an **infinite** long line is the **characteristic** impedance! So $Z_c = \sqrt{(z/y)}$ (1.19)
Z_c is independent of x, and we may therefore interprete this equation by saying that an impedance Z_c is presented to the line on the left of x, equal to the right of x. We may in circuit terms consider the possibility of providing a suitable component with impedance Z_c and using it to terminate a line. The part of the line between source and this load will then behave as though it were extending to infinity. The significance of the characteristic impedance Z_c is that it can replace an infinite length of line. A line terminated in the characteristic impedance is said to be *matched*.

We can suppose • a **lossy line**

Now we have to take into consideration $r, \mathcal{L}, \mathbb{C}$, neglecting g, so we get

$$\gamma \simeq \sqrt{(r+j\omega\mathcal{L})\, j\omega\mathbb{C}} = \alpha + j\beta \quad (1.20)$$

$$Z_c \simeq \sqrt{\frac{r+j\omega\mathcal{L}}{j\omega\mathbb{C}}} = \alpha' + j\beta' \quad (1.21)$$

with α: attenuation constant (nepers per metre)
β: propagation constant.

• a **lossless line**

so $r=g=0$.

Hence

$$\gamma = j\omega\sqrt{\mathcal{L}\mathbb{C}} \quad (1.22)$$

$$Z_c = \sqrt{\frac{\mathcal{L}}{\mathbb{C}}} \quad (1.23)$$

The phase-speed is determined by

$$v_{ph} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{C_o}{\sqrt{\mu_r \varepsilon_r}} \quad (1.24)$$

and because μ_r, $\varepsilon_r \geq 1$, the propagation speed in a cable is always $< C_o$.
The same is valid for the wavelength $\lambda = v_f/f < \lambda_o = C_o/f$.
Also valid for a lossless line

$$|\gamma| = \frac{\omega}{v_f} = \frac{2\pi f}{v_f} = \frac{2\pi}{\lambda} = k \qquad (1.25)$$

1.2 Reflections, impedance and VSWR

Let's consider a **lossless, finite** transmission line terminated in Z_L (load impedance).

1.2.1 Reflection coefficient (K,ρ,Γ)

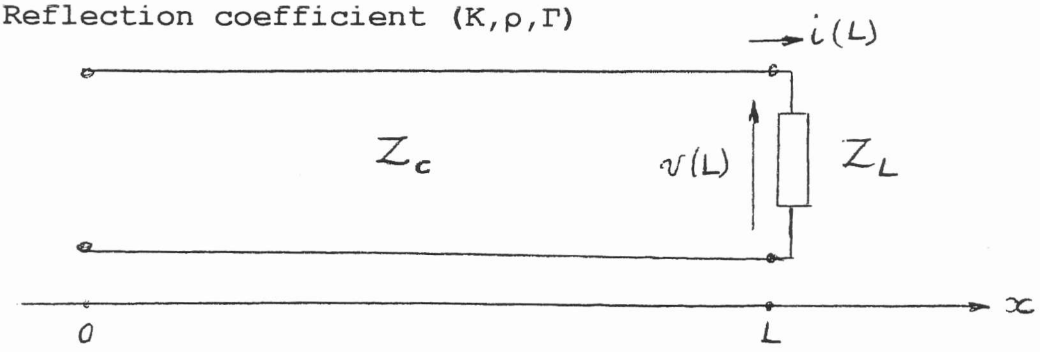


Fig.1-3: Line terminated in Z_L.

Whe consider a line terminated at x=L in an impedance Z_L.
With $\|\gamma\|=k$ we can write (1.14) as

$$v(L) = Z_L . i(L) = A_1 e^{jkL} + B_1 e^{-jkL} = \frac{Z_L}{Z_c}(-A_1 e^{jkL} + B_1 e^{-jkL}) \qquad (1.26)$$

This equation can be written as

$$A_1 e^{2jkL}(1+Z_L) = B_1(Z_L - 1) \quad so \quad A_1 = B_1 e^{-2jkL}\frac{Z_L - 1}{Z_L + 1} \qquad (1.27)$$

and call **$Z'_L = Z_L/Z_c$** the **normalised load impedance (to the characteristic impedance)**.
The term $(Z'_L-1)/(Z'_L+1)$ is defined as the reflection coefficient for the voltage at the load, called K (or Γ,ρ).
For the current at the load the reflection coefficient is -K.
The term e^{-2jkL} is the phase factor, only depending of the length L.

1.2.2 Impedance

Take again (1.14) and (1.15) in which we substitute A_1 by (1.27), we get

$$v(x) = B_1 . K . e^{-2jkL} e^{jkx} + B_1 e^{-jkx} \qquad (1.28)$$

$$i(x) = \frac{1}{Z_c}(-B_1 . K . e^{-2jkL} e^{jkx} + B_1 e^{-jkx}) \qquad (1.29)$$

The ratio $z(x)=v(x)/i(x)$ is called the *local impedance* at the point x, resulting in

$$z(x) = z_c \frac{Ke^{-2jkL} e^{jkx} + e^{-jkx}}{-Ke^{-2jkL} e^{jkx} + e^{-jkx}} \qquad (1.30)$$

We can write the normalised local impedance as:

$$z'(x) = \frac{z(x)}{z_c} = \frac{1+K(x)}{1-K(x)} \qquad (1.31)$$

$$\text{with } K(x) = Ke^{-2jk(L-x)} \qquad (1.32)$$

$K(x)$ is the voltage reflecting coefficient at the point x, between the reflected (backward) wave $A_1 e^{jkx}$ and the forward wave $B_1 e^{-jkx}$. Substituting K by $(Z'_L-1)/(Z'_L+1)$ in (1.32) and after substitution of (1.30) we can find for $z'(x)$

$$z'(x) = \frac{Z_L + jtgk(L-x)}{1+jZ_L . tgk(L-x)} \qquad (1.33)$$

Filling in x=0, we can find the (normalised) input impedance of a transmission line with load Z_L at x=L

$$z'_{in} = z'(o) = \frac{z_L + jtgkL}{1+jz_L tgkL} \qquad (1.34)$$

With (1.34) we can examine some special cases:

- input impedance of a short line ($L \ll \lambda$)
  $Z_{in} = Z_L$
- input impedance if $Z_L = Z_c$
  $Z'_{in} = Z_L/Z_c = 1$ (normalised)
  so $Z_{in} = Z_c$

- termination in an open-circuit

  $Z'_L = \infty$

  $Z'_{in} = -j.\text{cotg}(kL)$ (1.35)

  The input impedance (admittance) is imaginar.

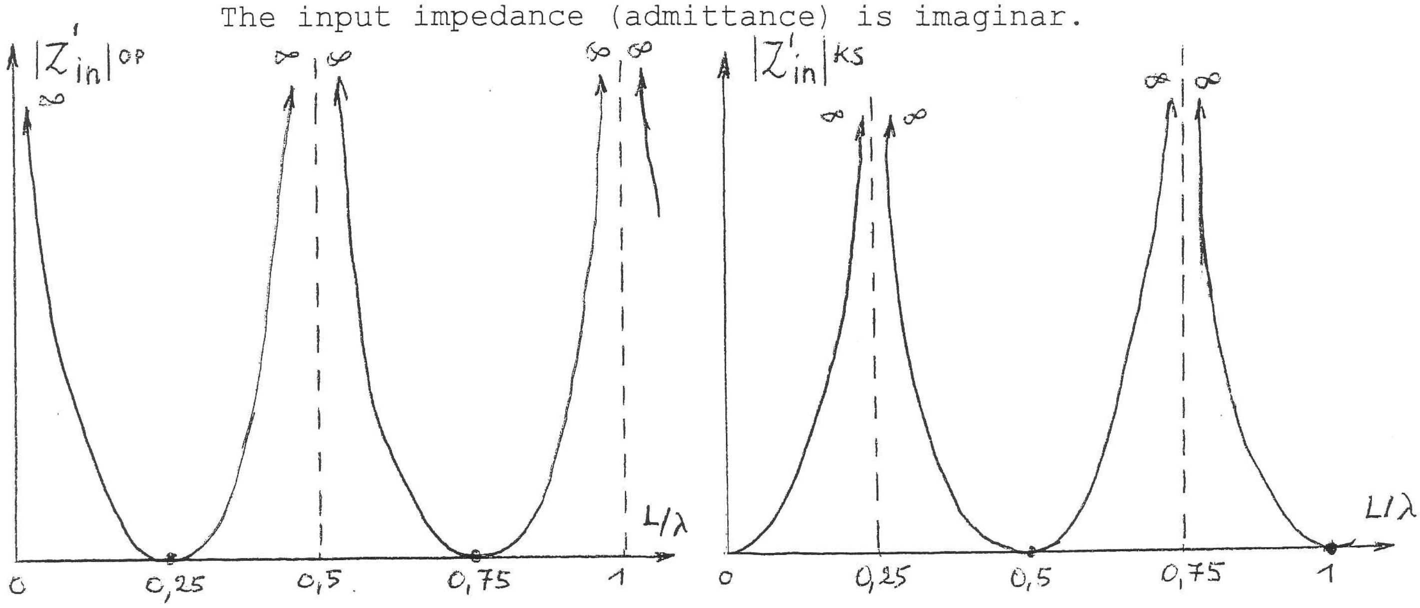


Fig.1-4: Input reactance for a length of line terminated in an open-circuit or a short-circuit.

Notice the pattern is periodic with $\lambda/2$!

The voltage reflection coefficient K=1. So at the point x=L the voltage doubles and the current will be 0.

- Termination in a short-circuit

  $Z'_L = 0$

  $Z'_{in} = j.\text{tg}(kL)$ (1.36)

  as shown in the previous figure.

  The input impedance is again imaginery, voltage refl. coeff. K=-1, so at the point x=L the current doubles and the voltage will be 0.

- input impedance of a $\lambda/2$ line

  $Z'_{in} = Z'_L$ (1.37)

  so the impedance measured at intervals of a half-wave length from the termination has the same value as Z_L !

  It seems that the $\lambda/2$ line has no effect on the configuration.

  The same for L = λ, $3\lambda/2$, 2λ, $5\lambda/2$, ...

- input impedance of a $\lambda/4$ line

$$z_{in} = \frac{1}{z'} \quad so \quad Z_{in} = \frac{Z_c^{\,2}}{Z_L} \qquad (1.38)$$

The line thus acts as an impedance inverter, we call it the **quarter-wave transformer**. The same for $L = 3\lambda/4$, $5\lambda/4$, ...

1.2.3 VSWR - Voltage Standing Wave Ratio

Thus the fraction of the incident voltage wave that is reflected by a

- line with a different characteristic impedance
  - load impedance, different from characteristic impedance,

is $K(x) = (z'(x)-1)/(z'(x)+1)$, derived from (1.31)

Let's examine K with regard to x, from

$$v(x) = A_1 e^{jkx} + B_1 e^{-jkx} \qquad (1.39)$$

$$\text{and } K(x) = \|K\| e^{-2jk(L-x)} = \frac{A_1 e^{jkx}}{B_1 e^{-jkx}} \qquad (1.40)$$

($K(x)$ is a periodical function with amplitude $\|K\|$ and period $\lambda/2$) we get

$$v(x) = B_1 e^{-jkx}(1+K(x)) \qquad (1.41)$$

Now we can interprete $v(x)$ as

- one wave propagating in +x direction
- but with variable magnitude $|B_1.(1+K(x))|$.

The maximum amplitude along the line is $B_1.(1+\|K\|)$, the minimum is $B_1.(1-\|K\|)$, with $\|K\| \leq 1$ for passive loads. It will in general be complex, hence

$$K = \|K\| e^{j\varphi} \qquad (1.42)$$

So, analysis of $\|1+K(x)\|$ gives us the max and min amplitude at the line, **with a period $\lambda/2$,** for what concerns the absolute value.

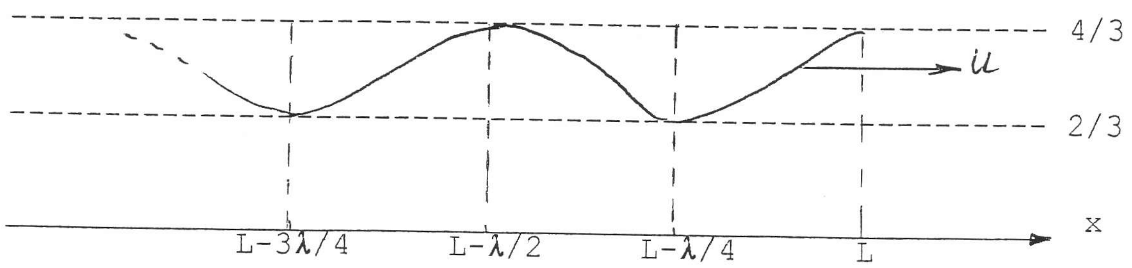


Fig.1-5: Standing-wave pattern.

Knowing the reflection coefficient, we may find the standing-wave ratio as

$$\frac{V_{max}}{V_{min}} = VSWR = \frac{1+|K|}{1-|K|} \qquad (1.43)$$

Further, we notice that the power, available in every point of a lossless line is defined as

$$P(x) = \frac{1}{2}Re(v(x).i^*(x)) \qquad (1.44)$$

$$= \frac{1}{2}[(A_1 e^{jkx}+B_1 e^{-jkx})(\frac{1}{Z_c})^*(-A_1^* e^{-jkx}+B_1^* e^{jkx})]$$

$$= \frac{1}{2}\frac{1}{Z_c}[\|B_1\|^2-\|A_1\|^2] \qquad (1.45)$$

So the power reflection coefficient is

$$\frac{\|A_1\|^2}{\|B_1\|^2} = \|K\|^2 \qquad (1.46)$$

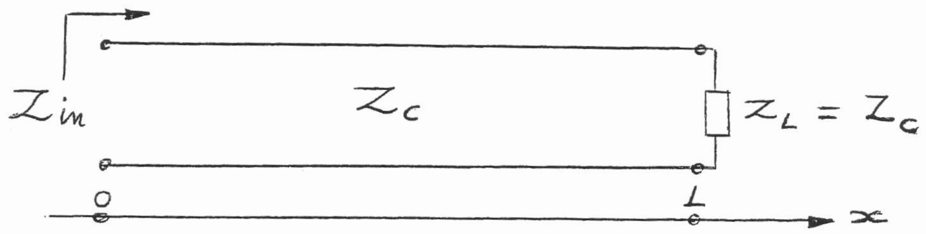
The relative power dissipated in the load can be written as

$$1-\|K\|^2 \qquad (1.47)$$

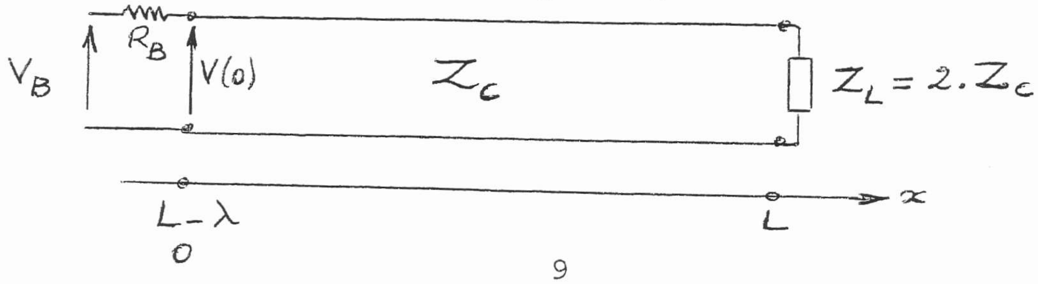
1.3 Voltage and current standing wave patterns
Examples with real and complex loads

Let's examine the wave pattern and the input impedance with some examples

- *example 1* : input impedance



- *example 2* : termination in $2.Z_c$, length L=λ



$$v(x) = |1+K(x)| = |1+\frac{1}{3}e^{-j2k(L-x)}|$$
$$= |1+\frac{1}{3}cos2k(L-x)-j\frac{1}{3}sin2k(L-x)|$$
$$= \sqrt{1+\frac{1}{9}\cos^2 2k(L-x)+\frac{1}{9}\sin^2 2k(L-x)+\frac{2}{3}cos2k(L-x)}$$

$$= \sqrt{\frac{10}{9}+\frac{2}{3}cos2k(L-x)}$$

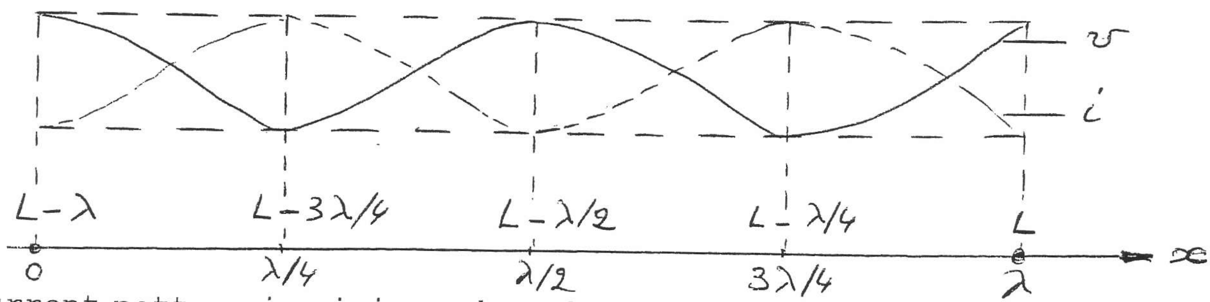
minimum:

$$cos2k(L-x) = -1, \; so$$
$$x_{min} = L-\frac{(2n+1)}{(2)}\frac{\lambda}{2}$$

The corresponding minimum is 2/3.
maximum:

$$\cos 2k(L-x) = 1, \; so$$
$$x_{max} = L-n\frac{\lambda}{2}$$

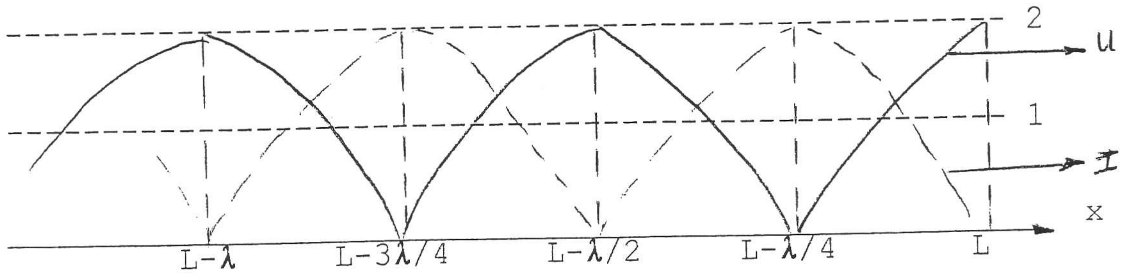
with maximum 4/3.
The max, 4/3, corresponds with E,
The min, 2/3, corresponds with E/2.
The *voltage* pattern becomes



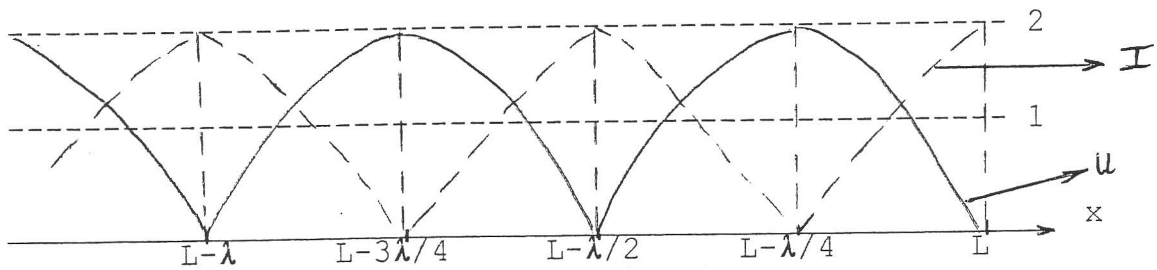
The current pattern is minimum when the voltage pattern is max, and max when the voltage pattern is min.
Hence for the *current* we can work with -K

• *example 3* : termination in open/short circuit

Standing-wave pattern for open-circuit:



Standing-wave pattern for short-circuit:



<u>1.4 Smith-chart</u>

An important addition to our analytical and design tool will be the use of a graphical technique for solving reflection and matching problems. Probably the most widely used one is the Smith-chart.
Basically, this diagram shows curves of constant resistance and constant reactance; these may represent either an input impedance or a load impedance.

1.4 Smith-chart

An important addition to our analytical and design tool will be the use of graphical technique for solving reflection and matching problems. Probably the most widely used one is the Smith-chart.
Basically, this diagram shows curves of constant resistance and constant reactance; these may represent either an input impedance or a load impedance.
We already studied

$$K_L = \frac{Z_L - Z_c}{Z_L + Z_c}$$

and

$$K(x) = K_L e^{-2jk(L-x)} \quad with \quad K_L = \|K_L\| e^{j\phi} \; and \; \|K_L\| \leq 1 \qquad (1.48)$$

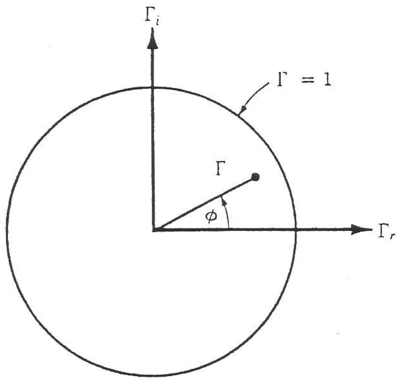


Fig.1-6: Polar coordinates of the Smith chart.

The diagram is constructed within a circle of unit radius, using polar coordinates. Peculiarly enough, the reflection coefficient itself will not be plotted on the final chart, for these additional contours would make the chart very difficult to read.
The impedances which we plot on the chart will be normalized with respect to the characteristic impedance.
Considering

$$z'(x) = \frac{1+K(x)}{1-K(x)} = normalised\ impedance$$
$$= R' + jX' \qquad (1.49)$$

K(x) is complex, represented by $u+j.v$, gives

$$z'(x) = \frac{1+u+jv}{1-u-jv} \quad and$$

$$R' = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad X' = \frac{2v}{(1-u)^2+v^2} \qquad (1.50)$$

After several lines of elementary algebra, we may write (1.49) and (1.50) in forms which readily display the nature of the curves on u (K_r) and v (K_i) axes,

$$(u-\frac{R'}{1+R'})^2+v^2 = \frac{1}{(1+R')^2} \qquad (1.51)$$

$$(u-1)^2+(v-\frac{1}{X'})^2 = \frac{1}{X'^2} \qquad (1.52)$$

The first equation describes a family of circles, where each circle is associated with a specific value of resistance r.
The second equation represents a family of circles defined by a particular value of x. See fig.1-7.

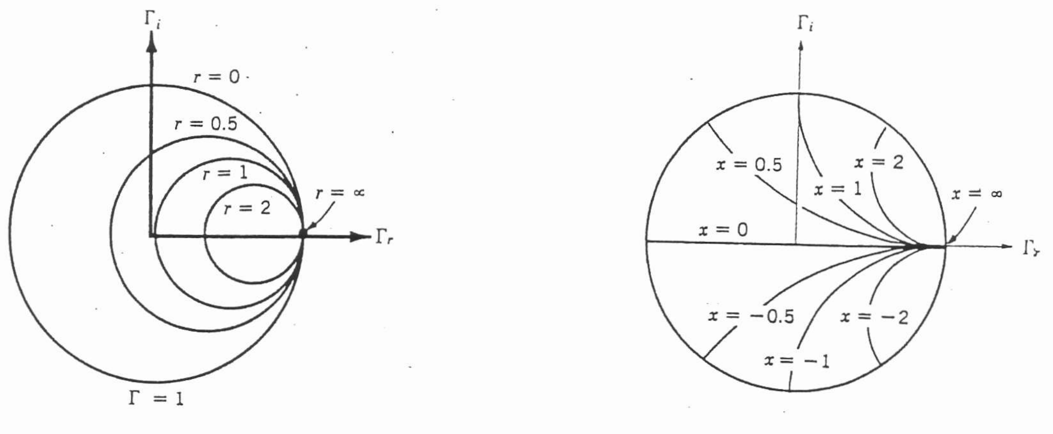


Fig.1-7: r and x circles on Smith chart.

It is now evident that if we are given Z_L, we may divide by Z_c to obtain Z', locate the appropiate r and x circles, and determine K by the intersection of the two circles. The angle of K is the counterclockwise angle from the K_r axis. A straight line from the origin through the intersection may be extended to the perimeter of the chart.

example : if Z_L=25+j50Ω on a 50Ω-line, Z'=0,5+j1, and point *A* on fig.1-8 shows the intersection of the r=0.5 and x=1 circles. The reflection coefficient is 0.62 at an angle of 83°.

The normalized input impedance produced by a normalized load impedance $z = 0.5 + j1$ on a line 0.3λ long is $z_{in} = 0.28 - j0.40$.

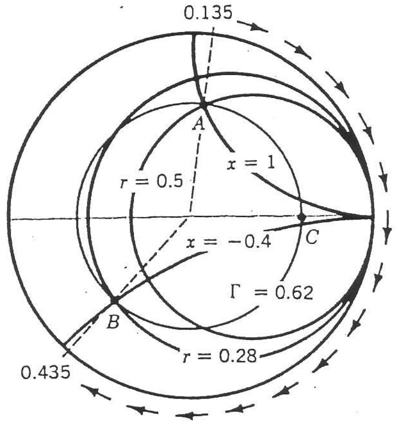
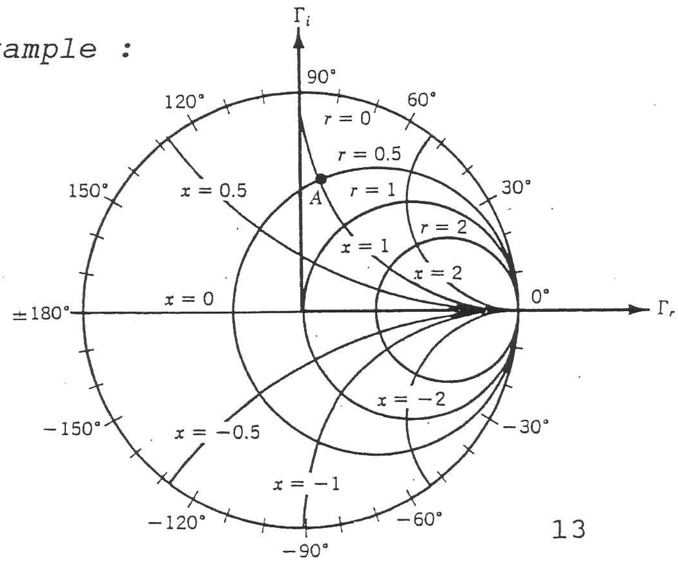


Fig.1-8: see example above.

The Smith chart is thus completed by the addition of a scale showing a change of 0.5λ for one circumnavigation of the unit circle. For convience, two scales are usually given, one showing an increase in distance for clockwise movement and the other an increase for counterclockwise travel. Note that the one marked "wavelength toward generator" (wtg) shows increasing values of $1/\lambda$ for clockwise travel, as described above. The zero point of the wtg scale is rather arbitrarly located to the left. This corresponds to input impedances having phase angles of 0° and $R_L<Z_o$. We have also seen that voltage minima are always located here.

- *example :*



· At the bottom of the chart we can find some extra scales. Let's illustrate them with the point $z'=2$. We can read on the scales on the right "voltage reflection coefficient": $|K|=0.33$, "power reflection coefficient": $|K|^2=0.11$, "return loss in dB": $-10\log |K|^2=9.54$dB, ...
At the left we can read the VSWR in voltage or dB and some extra scales for lossy lines.

· What about admittances? We know $Z'_L=Z_L/Z_c$ and $Y'_L=Z_c/Z_L=Y_L/Y_c$, so

$$Y'(x) = \frac{1-K(x)}{1+K(x)} = \frac{1+[-K(x)]}{1-[-K(x)]} \qquad (1.53)$$

With $-K(x)$ we can read $Y'(x)$ and we can find $-K(x)$ by reflection of $K(x)$ with respect to the point (0,0). So by reflection of $Z'(x)$ with respect to (0,0) we get immediately $Y'(x)$.

The (practical) use of the Smith chart is shown in the following chapters.

2. Transmission lines, pulse waveforms

We have been considering the transmission effects from the point of view of signals restricted to narrow bands of frequency. Digital signals consist of a sequence of pulses.
Now we're going to study the temporary transition-phenomena of transmission lines when applied with impuls voltages (step-function, Dirac, impuls waveforms, ...).
The same phenomena appear when we consider digital (logic) signals and send them with high datarate over a long distance.

2.1 Voltage and current equations for complex charged lines with an ideal step-function source (t_{rise}=0)

Consider an incremental length of a uniform transmission line.

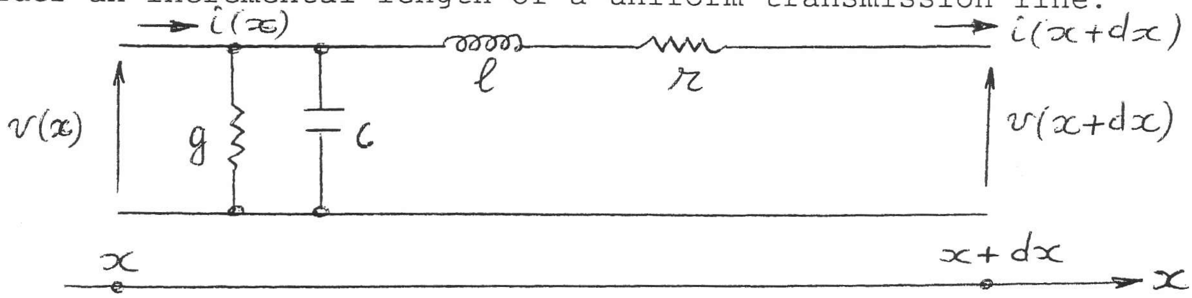


Fig.2-1: Circuit equivalent to elementary length of transmission line.

$$v(x+dx)-v(x) = -ri(x+dx)\,dx-\mathcal{L}\frac{\delta i}{\delta t}dx \qquad (2.1)$$

$$i(x+dx)-i(x) = -gv(x)\,dx-\mathbb{C}\frac{\delta v}{\delta t}dx \qquad (2.2)$$

Rewrite it as

$$\frac{\delta v}{\delta x} = -ri-\mathcal{L}\frac{\delta i}{\delta t} \qquad (2.3)$$

$$\frac{\delta i}{\delta x} = -gv-\mathbb{C}\frac{\delta v}{\delta t} \qquad (2.4)$$

When we suppose a lossless structure, then r=g=0:

$$\frac{\delta v}{\delta x} = -\mathcal{L}\frac{\delta i}{\delta t} \quad or \quad \frac{\delta^2 v}{\delta x^2} = \mathcal{L}\mathbb{C}\frac{\delta^2 v}{\delta t^2} \qquad (2.5)$$

$$\frac{\delta i}{\delta x} = -\mathbb{C}\frac{\delta v}{\delta t} \quad or \quad \frac{\delta^2 i}{\delta x^2} = \mathcal{L}\mathbb{C}\frac{\delta^2 i}{\delta t^2} \qquad (2.6)$$

The solution of these differential equations are can be written as

$$f(x-v_f.t) \quad and \quad g(x+v_f.t) \quad with \quad v_f=\frac{1}{\sqrt{\mathcal{L}\mathbb{C}}} \qquad (2.7)$$

With f a forward wave (+x-direction) and g a backward wave (-x-direction), so we can write

$$v = A_1.f(u)+B_1.g(w) \qquad (2.8)$$
$$i = A_2.f'(u)+B_2.g'(w) \qquad (2.9)$$
$$with \quad u = x-v_f.t$$
$$and \quad w = x+v_f.t$$

When we use these solutions in the wave-equations we get

$$A_1\frac{\delta f}{\delta u}+B_1\frac{\delta g}{\delta w} = -l(A_2\frac{\delta f'}{\delta u}(-v_f)+B_2\frac{\delta g'}{\delta w}.v_f) \qquad (2.10)$$
$$A_2\frac{\delta f'}{\delta u}+B_2\frac{\delta g'}{\delta w} = -c(A_1\frac{\delta f}{\delta u}(-v_f)+B_1\frac{\delta g}{\delta w}.v_f) \qquad (2.11)$$

After some algebra and integration (to u and w) we find

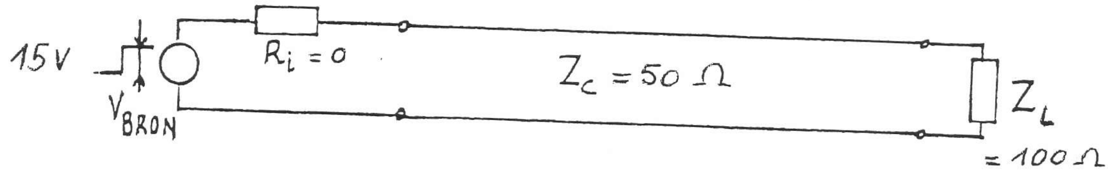
$$f = f'\frac{A_1}{A_2}\sqrt{\frac{\mathcal{L}}{\mathbb{C}}}+C_1 \quad and \quad g = g'(-\frac{B_2}{B_1})\sqrt{\frac{\mathcal{L}}{\mathbb{C}}}+C_2 \qquad (2.12)$$

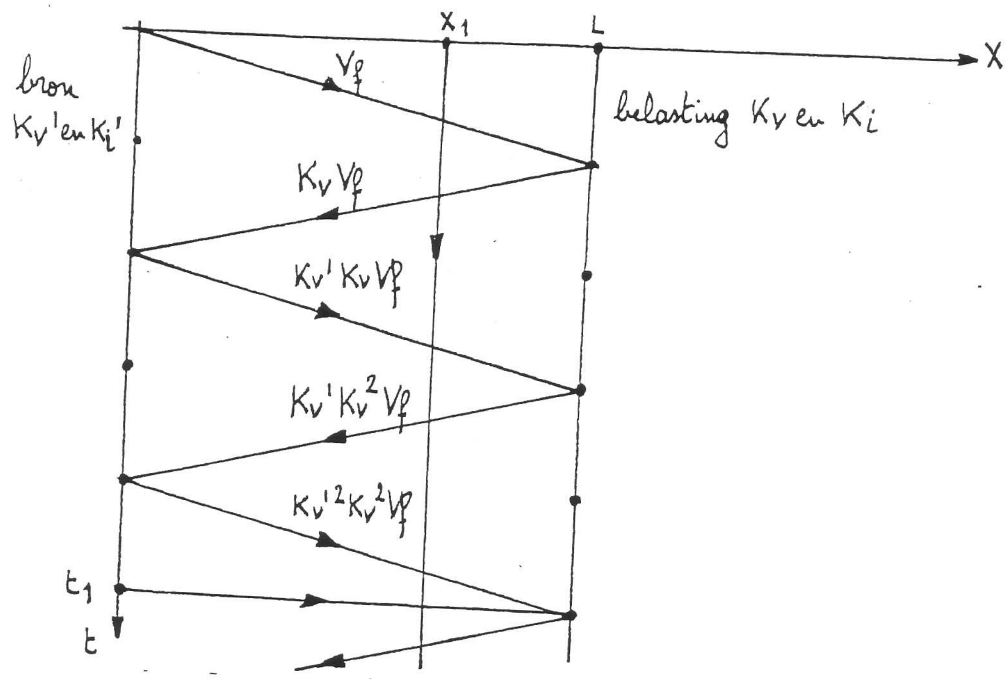
Because nor voltage nor current can't be zero alone (for each wave-front), we have with C_1 and C_2 =0 that voltage and current pattern are smilar, so

$$f = f' \quad and \quad g = g' \qquad (2.13)$$
$$A_2 = A_1\sqrt{\frac{\mathbb{C}}{\mathcal{L}}} \quad and \quad B_2 = -B_1\sqrt{\frac{\mathbb{C}}{\mathcal{L}}} \qquad (2.14)$$
$$with \quad \sqrt{\frac{\mathcal{L}}{\mathbb{C}}} \quad the\ characteristic\ impedance.$$

For a resistive load we can define a reflection coefficient as in the previous chapter.
For an **impuls the input 'impedance' of a transmission line is always the characteristic impedance!** So it's different from sinusoidal waves.

• *example 1 :* Line with switched input for surge calculation.





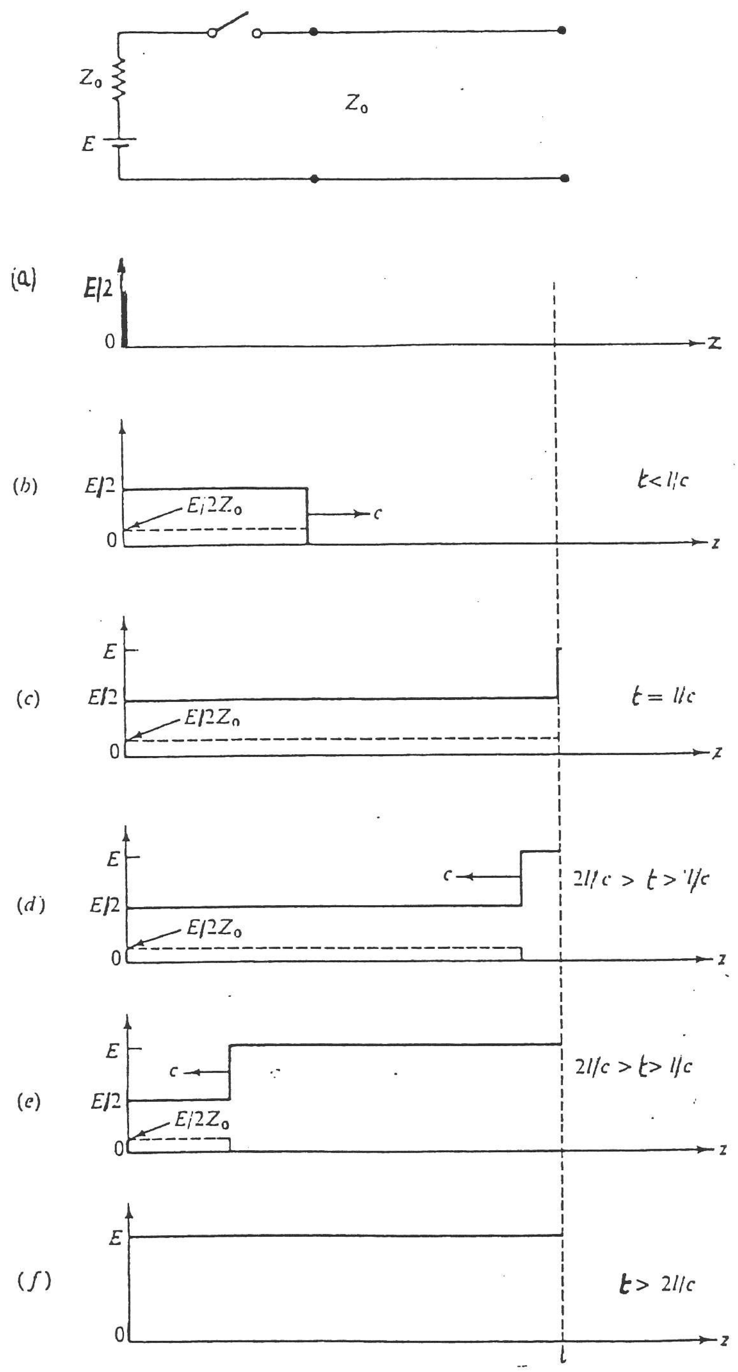
$$V(t_1, x_1) = V_f + K_v V_f + K_v' K_v V_f + K_v' K_v^2 V_f + K_v'^2 K_v^2 V_f = 15\ \mathrm{V}$$

$$I(t_1, x_1) = I_f + K_i I_f + K_i' K_i I_f + K_i' K_i^2 I_f + K_i'^2 K_i^2 I_f = 0{,}167\ \mathrm{A}$$

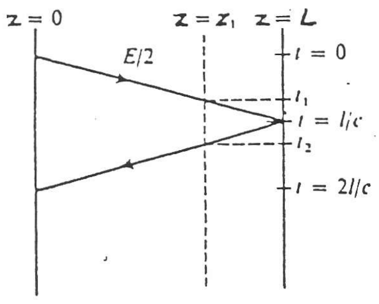
This process may be conveniently represented on a lattice diagram as shown in *example 2*. The initial surge ½E is travelling from the input. At the termination this surge is totally reflected and travels back along the line to the input. Thus at point z_1 the voltage will vary

with time as shown in the figure below. Since the reflected surge sees a correct termination at the input of the line, there is no further reflection, and the line remains charged to the voltage E.
Lattice diagrams can be used for more complicated problems on lossless lines involving multiple reflections.

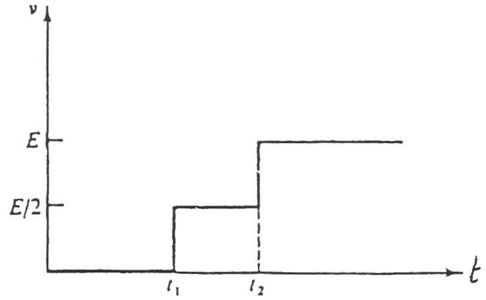
- *example 2 :* example 1 with lattice diagram



circuit with lattice diagram



(a)



(b)

What about a non-resistive load, temporary-transient phenomena?

Consider fig.2-2, assuming the line is **long** and **t=0** when the pulse arrives at the load.

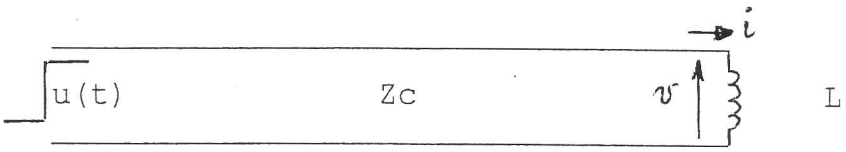


Fig.2-2: Line with inductive load.

We can always write

$$v = f+g$$
$$i = \frac{1}{Z_c}.(f-g) \qquad (2.15)$$

with f, the forward wave and g, the backward wave.
But, we need a 3_{th} equation with regard to our load. Here it is

$$v = \mathcal{L}.\frac{di}{dt} \qquad (2.16)$$

First we search g and then add f, so we have the voltage at the load.

$$f+g = \frac{\mathcal{L}}{Z_c}.\left(\frac{\partial f}{\partial t}-\frac{\partial g}{\partial t}\right)$$

$$g+\frac{\mathcal{L}}{Z_c}.\frac{\partial g}{\partial t} = -f+\frac{\mathcal{L}}{Z_c}.\frac{\partial f}{\partial t}$$

$$\text{so } g+\tau\frac{\partial g}{\partial t} = 0 \rightarrow g+D\tau g = 0$$

$$\text{with } D = -\frac{1}{\tau} \text{ (D-operator)}$$

$$\Rightarrow g(t) = C.e^{-\frac{t}{\tau}}.u(t)$$

$$f \leftrightarrow u(t)$$

$$\frac{du(t)}{dt} = \delta(t) \quad \text{(Dirac)}$$

$$g(t) = C.e^{-\frac{t}{\tau}}.u(t)+A.u(t)+B.\delta(t)$$

$$g+\tau\frac{\partial g}{\partial t} = -f+\tau\frac{\partial f}{\partial t}$$

$$\Rightarrow A=-1,\ B=0,\ C=2$$

so g(t) becomes

$$g(t) = (2.e^{-\frac{t}{\tau}}).u(t)$$

so the voltage at the load becomes v=f+g

$$v = 2.e^{-\frac{t}{\tau}}.u(t)$$

indeed, the voltage first sees an open circuit (v doubles) and then gets short (v=0).

The current at the load becomes

$$i = \frac{1}{Z_c}.(f-g)$$

$$= \frac{2.u(t)}{Z_c}.(1-e^{-\frac{t}{\tau}})$$

indeed, the current first sees an open circuit (i=0) and then gets short $(2u/Z_c)$.

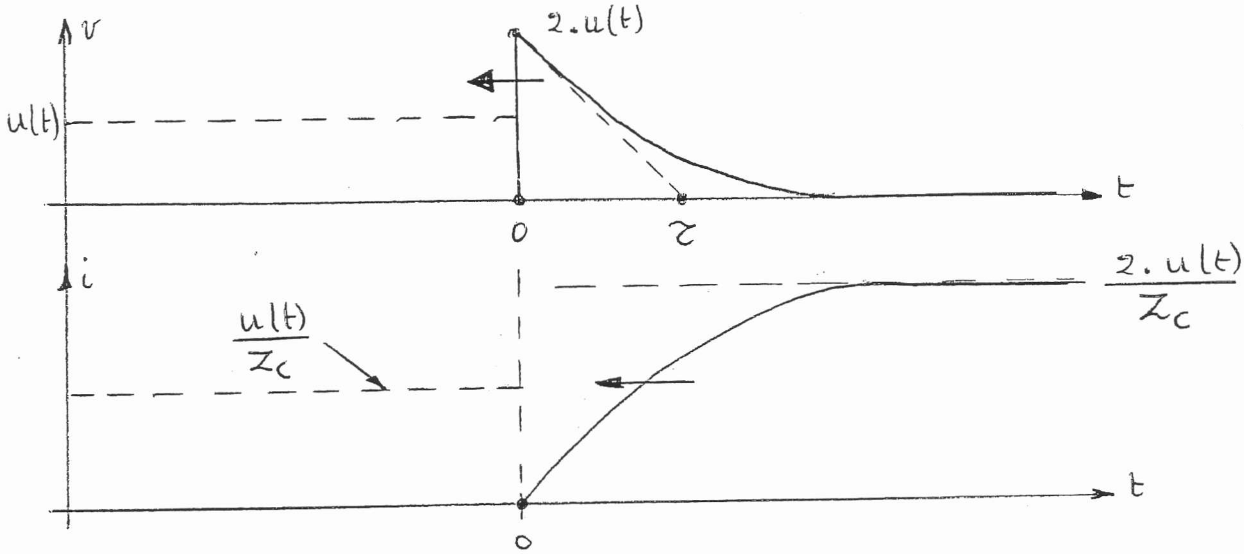


Fig.2-3: Voltage and current at the load (t=0).

2.2 Voltage and current equations for complex charged lines with a practical step-function source ($t_{rise} \approx 0$)

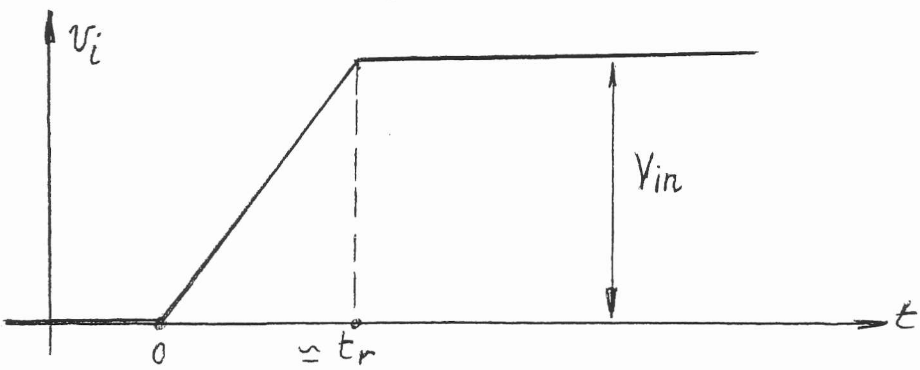


Fig.2-4: Impuls with $t_{rise} \neq 0$. Possible circuit.

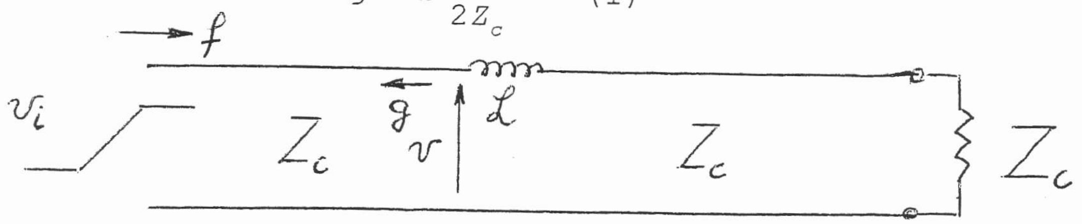
We suppose that $t_{rise} \neq 0$, so we can associate a maximal frequency with it. Say t_{rise} is large enough (or $\mathcal{L}$ small enough) so $\omega\mathcal{L} << Z_c$ over the complete frequency spectrum.
The reflection coefficient at the inductance $\mathcal{L}$ is

$$K = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{j\omega\mathcal{L}}{2.Z_c + j\omega\mathcal{L}} = \frac{V_{refl}}{V_{source}} = \frac{g}{f}$$

$$\approx \frac{j\omega\mathcal{L}}{2.Z_c}$$

So

$$g = f\frac{j\omega\mathcal{L}}{2Z_c} \qquad (1)$$



The waveform f (applied source) contains a finite number of frequencies, and (1) contains the variable ω. Because

$$j\omega f = \frac{d}{dt}f \qquad (2)$$

$$with \quad f = V_{in}e^{j\omega t} \qquad with\ V_{in} = amplitude\ of\ f$$

for every frequency component must be

$$g = f\frac{j\omega\mathcal{L}}{2Z_c}$$

$$(2)\ in\ (1)\ gives\ g = \frac{\delta f}{\delta t}\left(\frac{j\omega\mathcal{L}}{2Z_c}\right)\frac{1}{j\omega} = \frac{\mathcal{L}}{2Z_c}\frac{\delta f}{\delta t}$$

Because the peak of the reflected wave g has to be proportional with the maximum slope of the t_{rise} we have

$$g = \frac{\mathcal{L}}{2Z_c}\left(\frac{\delta f}{\delta t}\right)_{max}$$

$$with \left(\frac{\delta f}{\delta t}\right)_{max} = \frac{V_{in}}{t_r}$$

$$so\ g = \frac{\mathcal{L}}{2Z_c}\frac{V_{in}}{t_r}$$

The amplitude of the reflected wave gets reduced because of the finite t_{rise}.

Further the reflected impuls is spreaded out

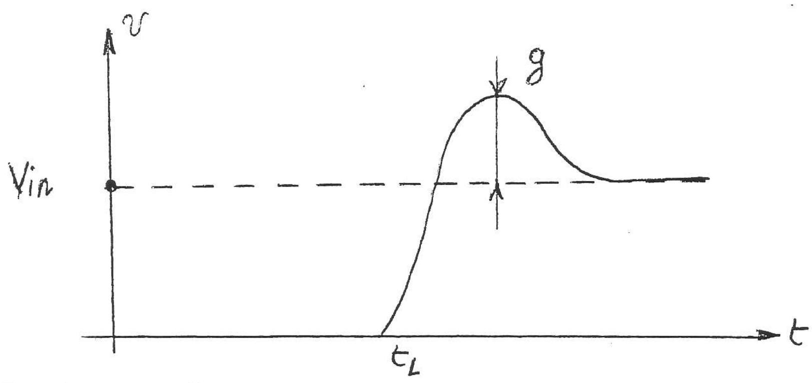


Fig.2-5: $t_{rise} \neq 0$

While an ideal step-function gives

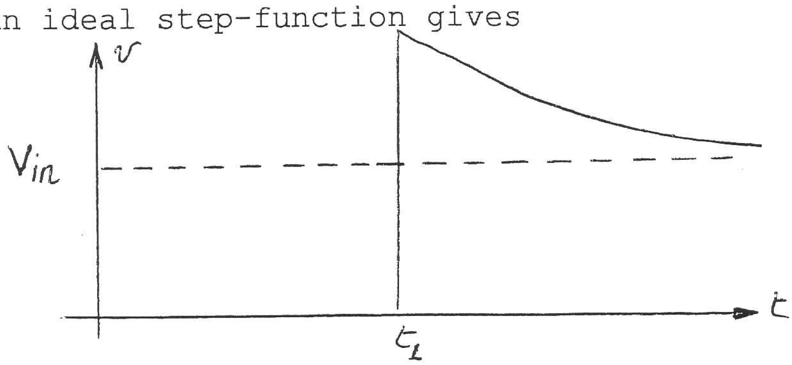


Fig.2-6: $t_{rise} = 0$

2.3 Non-linear loads - Bergeron method

So far the load was supposed to be linear. With the Bergeron method we can also work with non-linear loads. To illustrate this we give the following example

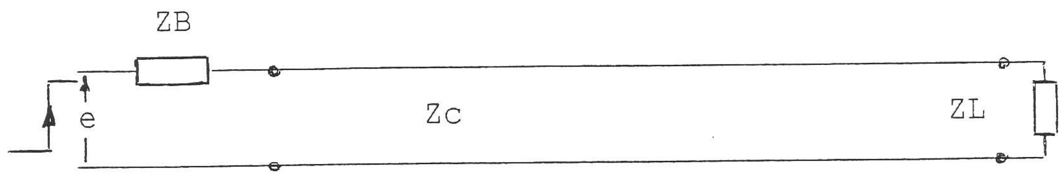


Fig.2-7 : Transmission line with non-linear load.

We draw the load-line Z_B, de load-caracteristic Z_L and the (straight) line with slope Z_C.

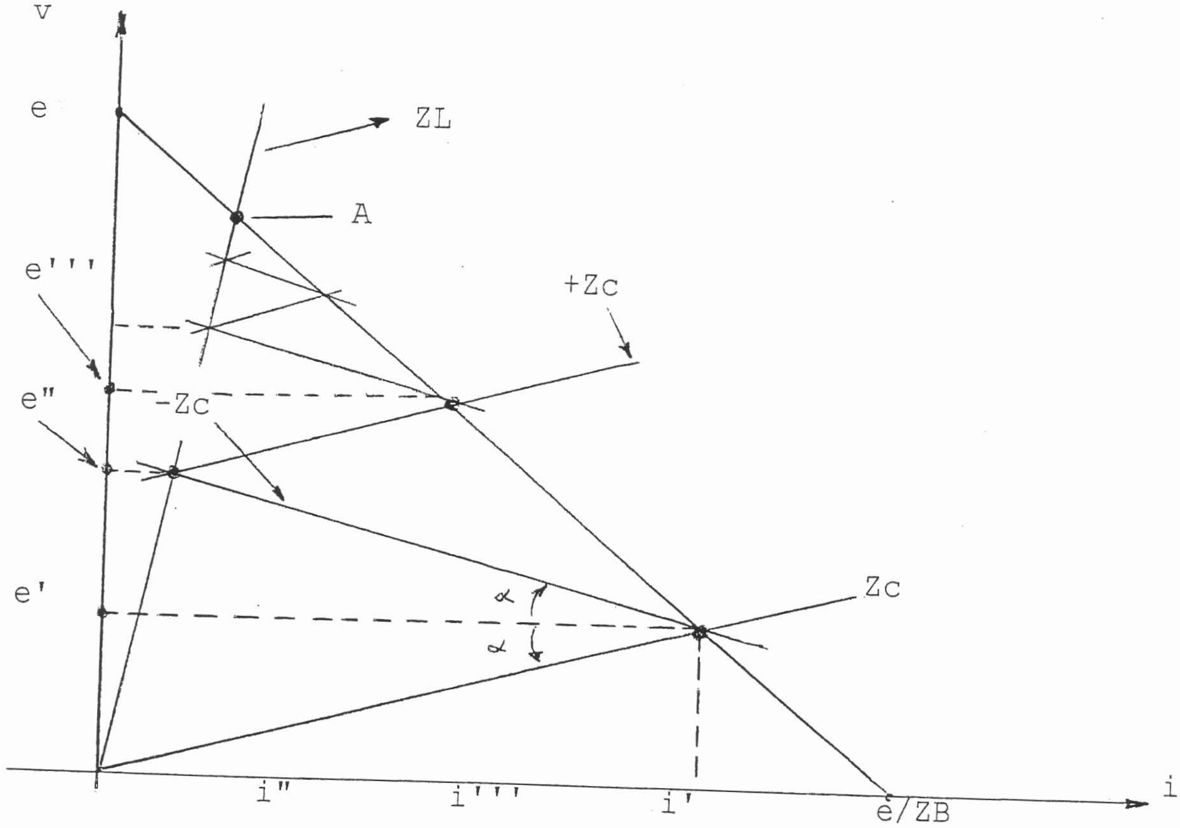


Fig.2-8 : v/i-plot with Bergeron method.

The (effective) voltage on the line is e'

$$e' = e.\frac{Z_C}{Z_C+Z_B}$$

The current i' starts to flow

$$i' = \frac{e}{Z_C+Z_B}$$

These co-ordinates are found on the point of intersection of the load-line Z_B and Z_C.

When the (forward) wave arrives at the load, there will be a reflection, so at Z_L there's a voltage e", corresponding with a current i"

$$e'' = (1+K).e' = (1+\frac{Z_L-Z_C}{Z_L+Z_C}).e' = \frac{2Z_L}{Z_L+Z_C}.e'$$

Constructing a (straight) line with slope $-Z_C$ in the point e', then gives the intersection between those line and the line of Z_L the point (e",i").

Indeed, the value e" in the plot is equal to

$$e'' = e' + Z_C(i' - i'') \text{ with } e' = Z_C . i' \text{ and } e'' = Z_L . i''$$

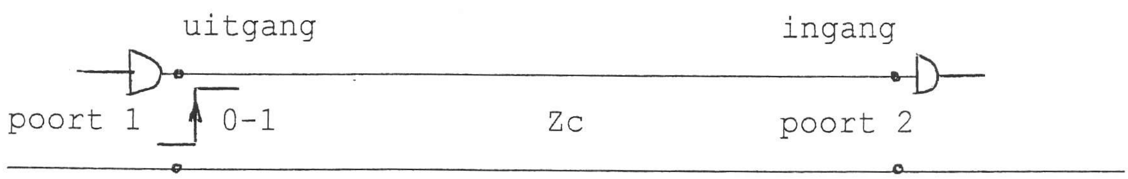
$$\text{so } i'' = \frac{2e'}{Z_L + Z_C} \text{ and } e'' = \frac{2e'Z_L}{Z_L + Z_C}$$

When the (reflected) wave arrives back at the source, we'll have the voltage e"'. So drawing a (straight) line with slope $+Z_C$ from e", we can find e"' as the intersection between this line and the source-caracteristic Z_B (load-line).
Whe can interpret this as: the reflected wave e" (direction of the source) acts as a 'source' with internal resistor Z_C.
But the direction of the current is opposite to these one of the convention as load-caracteristic, so whe have a slope $+Z_C$.
When whe consider a sequence of reflections, whe notice that there is a convergention to the point *A*. This is normal, because when the transition-phenomena are passed, whe have a DC-source (with internal resistor Z_B) and load Z_L. The voltage and current will be

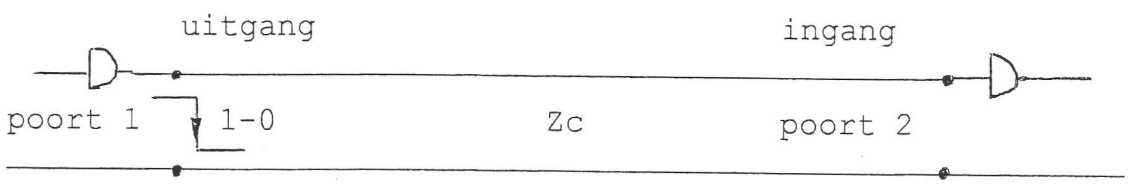
$$V_A = e . \frac{Z_L}{Z_L + Z_B} \qquad i_A = \frac{e}{Z_L + Z_B}$$

The disadvantage of this method is there is no respect to time.
The advantage is that non-linear source and/or load-caracteristics can be worked out.

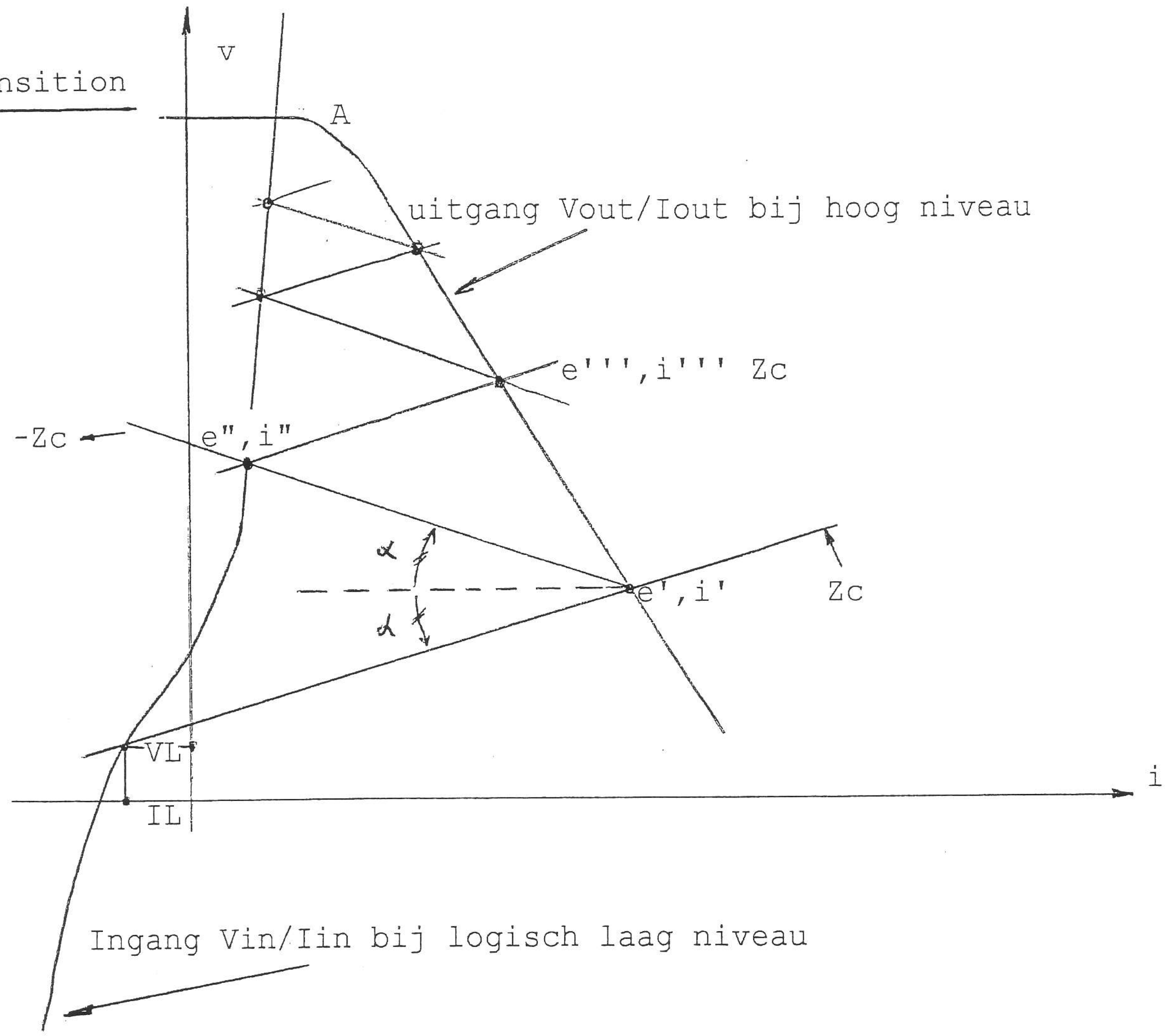
example : TTL-circuit with 0→1 transition



example : TTL-circuit with 1→0 transition



0→1 transition



1→0 transition

v

A(VH,IH)

e',i'
Zc

-Zc e"",i""

+Zc

i

B

e"',i"'

e",i"

-Zc

Vout/Iout
uitgang 1 LOW

Vin/Iin ingang 2

The coupling of semi-conductors on long lines is a critical work. The output stage of TTL/ECL technology is always an emitter-follower or totem-pole output (possible open-collector).
Totem-pole outputs give a low-impedant 'source' at 1 (Q_1 and D_1 conduct), so $Z_{out} \approx 0$ and parasitic capacitors have low influence (avoid open-collector!). Also ECL with emitter-follower gives low output-impedance. We can built high datarate-circuits.
We have to match the line but to restrict losses we avoid the following circuit.

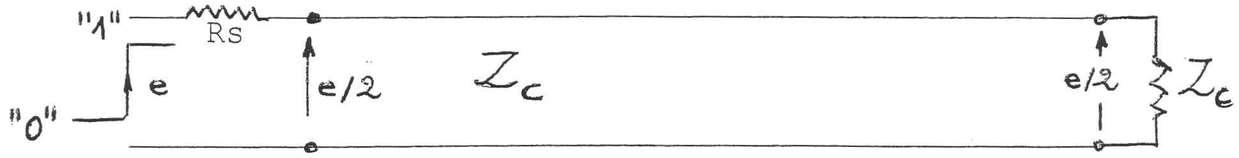


Fig.2-9: Matched line with Z_C resistor.

Ro = 5à7 Ohm.

A B C

Ro

Zc

Re

-VEE=-5,2V

Fig.2-10: Direct-coupling on line with (wanted) reflections ECL.

Coupling of digital systems prefers ECL-circuits because of have high input-impedance.

Bij ECL is de zwaai 0,9V (-0,9V = "1" ; -1,8V = "0"). We beschouwen een 1-0 gevolgd door een 0-1 transitie. Verder is tr (stijgtijd) << looptijd. We stellen dat Rs=68 en Zc=75 Ohm

VA

0,9V=ΔV ΔV

VB

ΔV/2 ΔV/2

ΔV/2 ΔV/2

VC

ΔV ΔV

tl tl tl tl

Fig.2-11: Coupling with ECL, waveforms.

3. Examples of transmission lines

3.1 Microstrip and stripline

These structures act as transmission lines, the caracteristic impedance is function of the dimensions (width of the strip, thickness of the dielectricum).
The development of solid-state microwave devices has led to very considerable use of transmission lines based on plane parallel conductors. These consist of a substrate which supports thin copper electrodes on either side, with the substrate thickness of the order of 1 mm. Some different forms of construction are shown in fig.3-3.

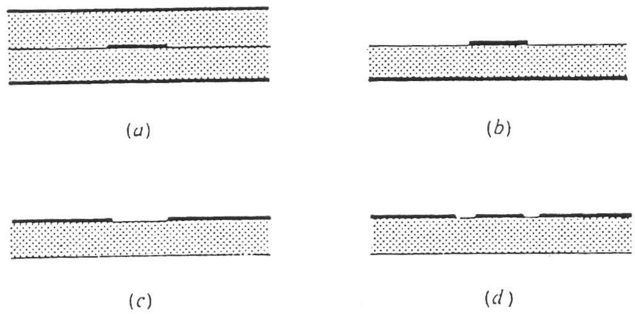


Fig.3-3: Forms of microstrip line.(a) Stripline.(b)Microstrip (c)Slot line.(d)Coplanar waveguide.

3.1.1 Stripline

See fig.3-3(a). This is the most difficult to fabricate, since it involves a sandwich of two planar substrates. It is however a true shielded transmission line using the outer planes as the shield and the middle strip as the inner. The interior is filled with dielectric.

$$Z_C = \frac{60}{\sqrt{\epsilon_r}} . \ln\left(\frac{4b}{0.67 . \pi . w(0.8+\frac{t}{w})}\right) \qquad (3.1)$$

$$\lambda = \frac{10^9}{3.336 f\sqrt{\epsilon_r}} \quad (m) \qquad (3.2)$$

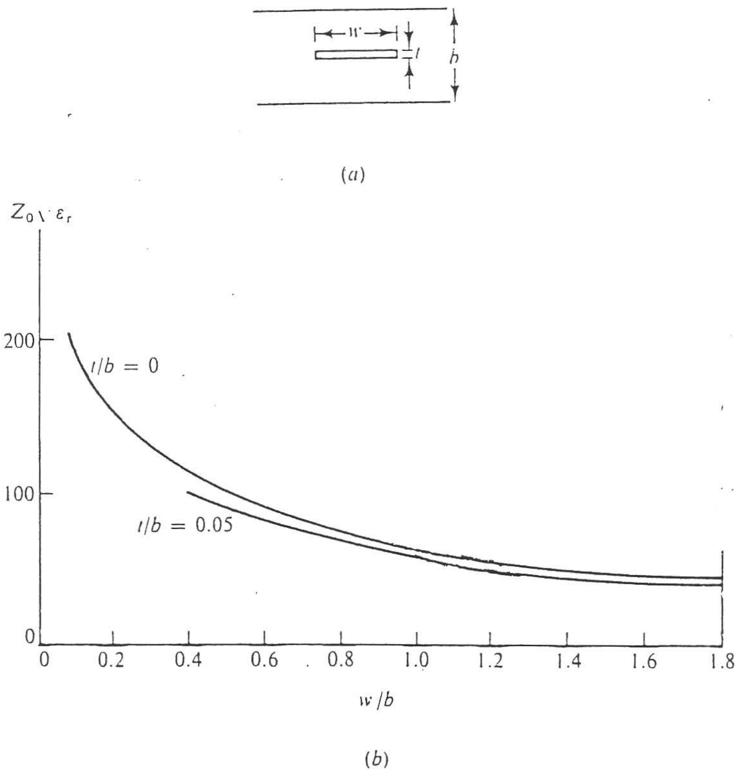


Fig.3-4: Characteristics of stripline.(a)Dimensions.(b)Z_c

3.1.2 Microstrip

The microstrip line of fig.3-3(b) is much easier to fabricate, although it is more lossy and is also dispersive.
This can be understood from the configuration of the lines of electric force as indicated in fig.3-5.

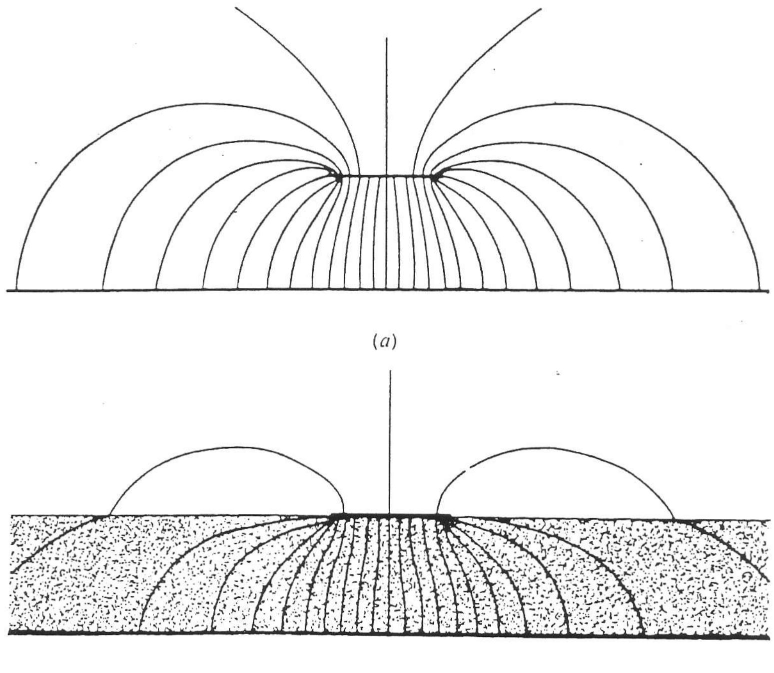


Fig.3-5: The effect of substrate permittivity.(a)Single-dielectric.(b)High-permittivity substrate.

The former shows the situation in single-dielectric, the latter the situation with a high-permittivity dielectric substrate.
In calculating the characteristic impedance, it is customary to introduce an effective permittivity, ϵ_{re}, which will be between the permittivity of the substrate and the surrounding air. The characteristic impedance and phase velocity of the actual line have the values which would be found for the conductor geometry immersed in an infinite dielectric of permittivity ϵ_{re}.

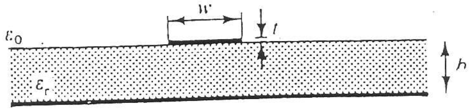


Fig.3-6: Dimensions of microstrip line.

For *0.1 < t/w < 3* we have

$$Z_C = \frac{87}{\sqrt{\epsilon_r+1.41}}\ln\left(\frac{5.98h}{0.8w+t}\right) \qquad (3.3)$$

$$\lambda = \frac{10^9}{3.336f\sqrt{0.475\epsilon_r+0.67}} \qquad (3.4)$$

For *t/w << 0.1* we can apply the Wheeler formules-graphs. Depending on the width of the strip, we have

$$\bullet w/h>2 \qquad Z_C=\frac{188.\frac{3}{\sqrt{\epsilon_r}}}{\frac{w}{2h}+0.441+\frac{\epsilon_r+1}{2\pi\epsilon_r}\left(\ln\left(\frac{w}{2h}+0.94\right)+1.451\right)+\frac{0.082(\epsilon_r-1)}{\epsilon_r^2}} \qquad (3.5)$$

$$\bullet w/h<2 \qquad Z_C=\frac{60}{\sqrt{\frac{\epsilon_r+1}{2}}}\left[\ln\left(\frac{8h}{w}\right)+\frac{1}{32}\left(\frac{w}{h}\right)^2-\frac{1}{2}\frac{\epsilon_r-1}{\epsilon_r+1}\left(0.451+\frac{0.241}{\epsilon_r}\right]\right. \qquad (3.6)$$

Because of the complexity of these formules, there are a lot of graphs available like fig.3-7.

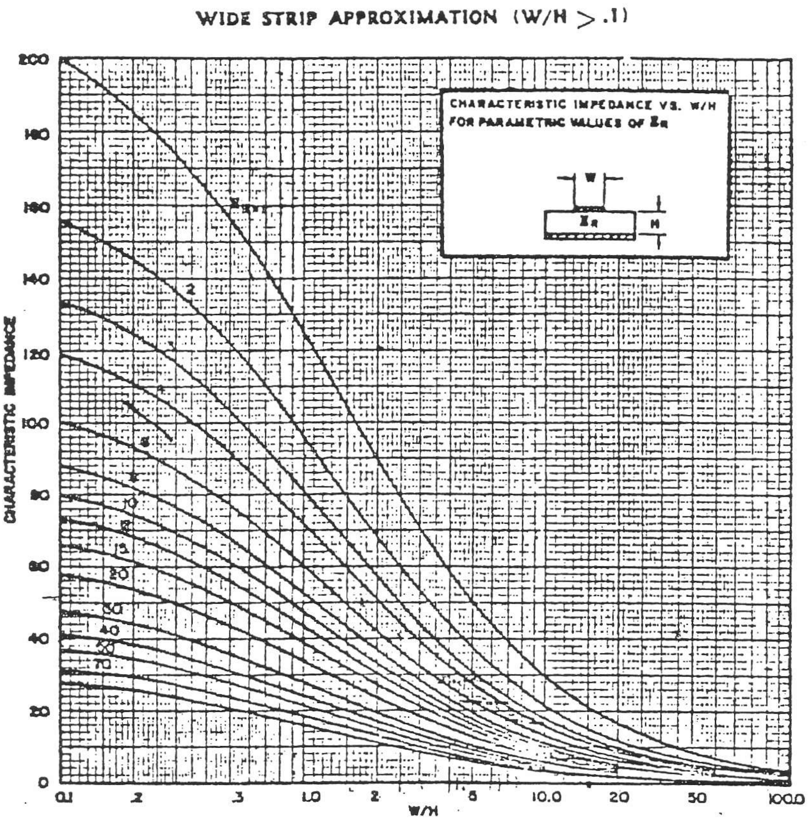


Fig.3-7: **Wheeler-graph** for $Z_c(w/h)$.

How creating a strip (w/h) with a wanted Z_c?
Determine ϵ_r of your substrate and read *w/h*. In literature there are formules (Hammerstädt) which gives now ϵ_{re} $[Z_c(\epsilon r), w/h]$. Calculate ϵ_{re}, which is useful to determine the actual wavelength on the strip

$$\lambda = \frac{c}{\sqrt{\epsilon_{re}} f}$$

Read **again** *w/h* on the Wheeler-graph.

For the determination of ϵ_{re} whe use the following Hammerstädt formules

$$\frac{W}{h}>1: \quad \epsilon_{re}=\frac{1}{2}(\epsilon_r+1)+\frac{1}{2}(\epsilon_r-1)\left(\frac{1}{\sqrt{1+12\frac{h}{W}}}\right) \qquad (3.7)$$

$$\frac{W}{h}<1: \quad \epsilon_{re}=\frac{1}{2}(\epsilon_r+1)+\frac{1}{2}(\epsilon_r-1)\left[\left(\frac{1}{\sqrt{1+12\frac{h}{W}}}\right)+0.04\left(1-\frac{W}{h}\right)^2\right] \qquad (3.8)$$

Some common substrates may be mentioned:
alumina, a ceramic substrate $\epsilon_r \approx 9.5$ to 10
epoxy (PCB) $\epsilon_r \approx 5$ to 6
polyguide, copper-clad irradiated plastic (polyolefin) $\epsilon_r \approx 2.3$.

The other types of strip line illustrated in fig.3-3 have properties similar to those discussed for microstrip line, and are covered in the literature.

3.1.3 Use of stripline and microstrip

A very important thing we can do with these strips is the creation of components (inductance, capacitor, 50Ω-line,...) for high frequencies.
In the first chapter we have seen

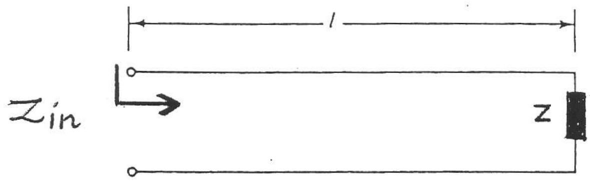


Fig.3-7: Z_{in} of transmission line.

$$Z_{in} = Z_C.\frac{Z'+j.tgkL}{1+j.Z'.tgkL} = Z_C.\frac{Z+j.Z_C.tgkL}{Z_C+j.Z.tgkL}$$

- *INDUCTANCE Z=0*

  suppose $L<\lambda/12$ so $tgkL \approx kL$
  if Z=0, and/or $Z<<Z_c.kL$ and $Z.kL<<Z_c$, then

$$Z_{in} = Z_C.\frac{j.Z_C.kL}{Z_C} = j.Z_C.kL = j.Z.\left(\frac{2.\pi}{v_f}\right).f.L$$

$$= j.2\pi f.\mathcal{L} \quad with \quad \mathcal{L} = \frac{Z_C.L}{v_f} = \frac{Z_C.L.\sqrt{\epsilon_{eff}}}{c} \qquad (3.9)$$

Now we have 2 parameters: L(length) and Z_c.

We also know

$$Z_C = \sqrt{\frac{\mathcal{L}}{C}}$$

so we will chose always Z_c **high-impedant for inductances**, with respect to the system-impedance 50Ω,
for example $Z_c=100\Omega$ (high to 50Ω) and let it terminate in a low-impedant load, for example $<10\Omega$ (low to 50Ω).
The length L of the line now determines $\mathcal{L}$.

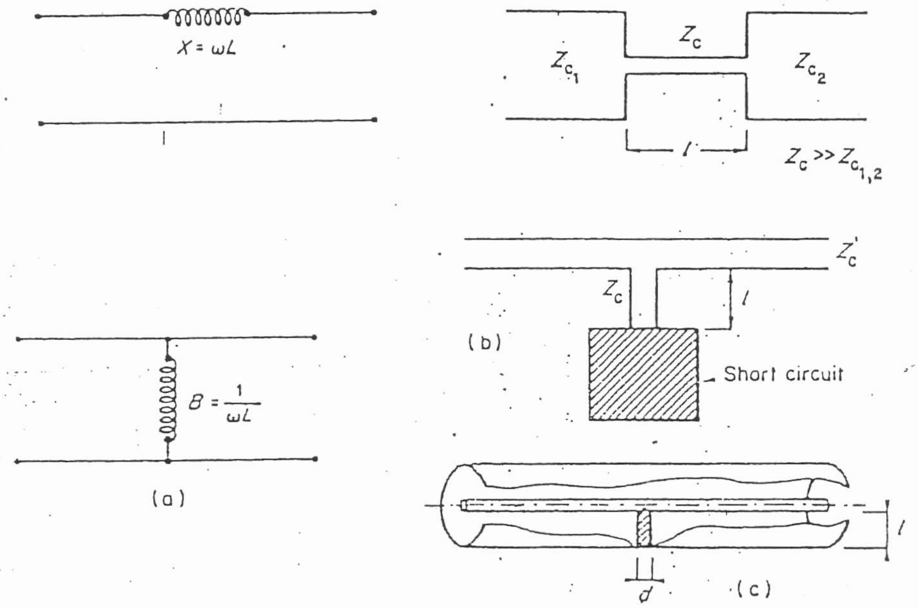


Fig.3-8: Examples of series and parallel inductances, created with microstrip transmission lines.

- *CAPACITOR* $Z=\infty$

  suppose $L<\lambda/12$ so $tgkL \approx kL$
  if $Z=\infty$, and/or $Z>>Z_c.kL$ and $Z.kL>>Z_c$, then

$$Z_\epsilon = \frac{Z_C}{j.kL} = \frac{Z_C.V_f}{j.2\pi f.L} = \frac{1}{j.\omega.\mathbb{C}}$$

$$\mathbb{C} = \frac{L}{Z_C.V_f} = \frac{L.\sqrt{\epsilon_{eff}}}{Z_C.c} \qquad (3.10)$$

We will chose always Z_c low-impedant for capacitors, for example $ZC=10\Omega$ and let it terminate in high-impedant loads, for example $>100\Omega$. The length of the line now determines $\mathbb{C}$.

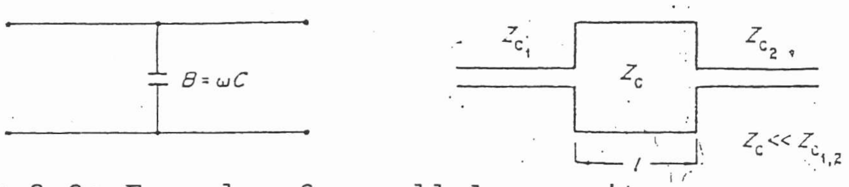


Fig.3-9: Example of parallel capacitor.
The creation of a **series capacitor is impossible** !!!!

Examples of some filters:

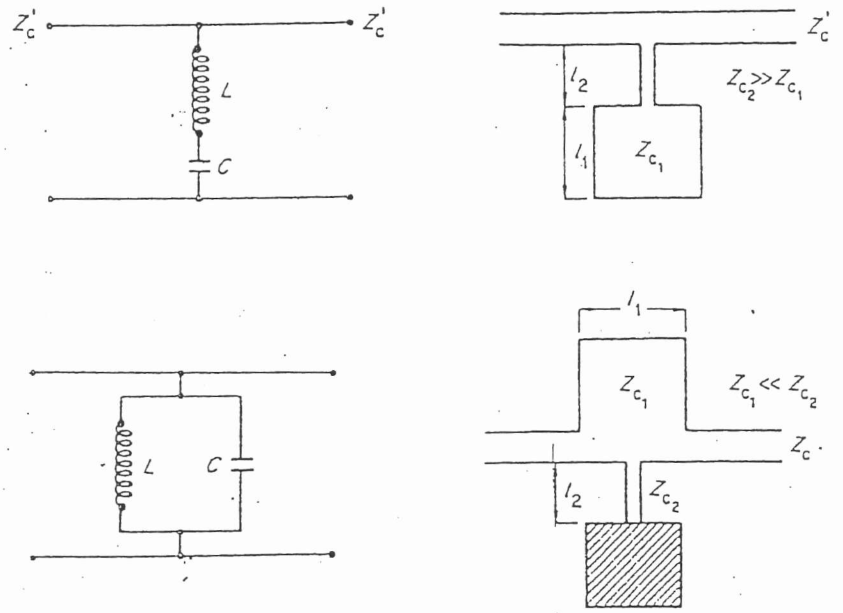


Fig.3-10: Resonant and anti-resonant circuits shunting the principal line.

The difficulty just mentioned means that the only resonant circuits that can be produced from line sections are resonant or anti-resonant circuits **shunting** the principal line.
The only way of placing one of these types of circuit in series with the principal line is to use the impedance-inverting property of a quarter-wave line. A well-known result from transmission line theory is that the input impedance Z_{in} of a quarter-wave line of characteristic impedance Z_c with a load Z_L is given by $Z_{in}=Z_c^2/Z_L$.

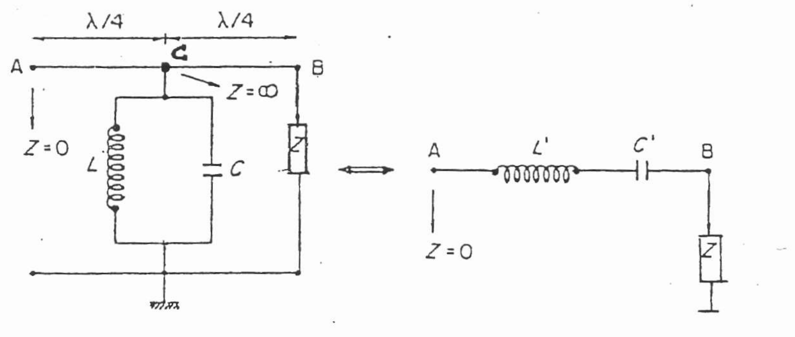


Fig.3-11: A parallel resonant circuit placed across a line between two quarter-wave sections is equivalent to a series resonant circuit placed along the principal line.

At point C we get

$$Z_{1,\parallel} = \frac{s\mathcal{L}\cdot\frac{1}{s\mathbb{C}}}{s\mathcal{L}+\frac{1}{s\mathbb{C}}} = \frac{s\mathcal{L}}{s^2\mathcal{L}\mathbb{C}+1}$$

$$\text{so } Y_{1,\parallel} = \frac{s^2\mathcal{L}\mathbb{C}+1}{s\mathcal{L}}$$

$$Z_{2,load} = \frac{Z_C^2}{Z} \quad \text{so} \quad Y_{2,load} = \frac{Z}{Z_C^2}$$

$$Y\ (\text{at point } C) = Y_1+Y_2$$

$$Z_{in,\ \parallel\ circuit} = Z_C^2 \cdot Y = Z+\left(\frac{s^2\mathcal{L}\mathbb{C}+1}{s\mathcal{L}}\right)\cdot Z_C^2$$

$$Z_{in,\ series\ circuit} = Z+\frac{1}{s\mathbb{C}'}+s\mathcal{L}'$$

Now Z_{in} of the series-circuit has to be equal to Z_{in} of the parallel-circuit. We obtain

$$\mathcal{L}' = \mathbb{C}\cdot Z_C^2 \rightarrow \mathbb{C} = \frac{\mathcal{L}'}{Z_C^2}$$

$$\mathbb{C}' = \frac{\mathcal{L}}{Z_C^2} \rightarrow \mathcal{L} = Z_C^2\cdot\mathbb{C}' \qquad (3.11)$$

Similarly, a series resonant circuit shunting the principal line between two quarter-wave lines (fig.3-12), is equivalent to a parallel resonant circuit placed along the principal line.

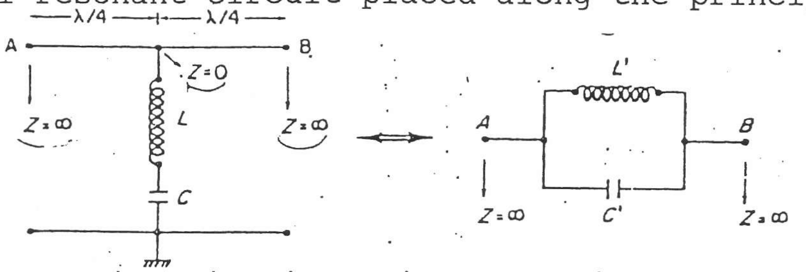
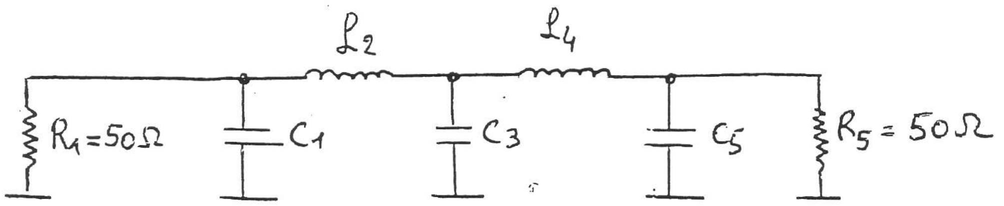


Fig.3-12: Series circuit equivalence with parallel circuit.

• *example:* Low-pass filter 5^{th} order, f_p=500MHz, Butterworth implementation, in-output-impedance 50Ω.

After some Butterworth-calculations we get

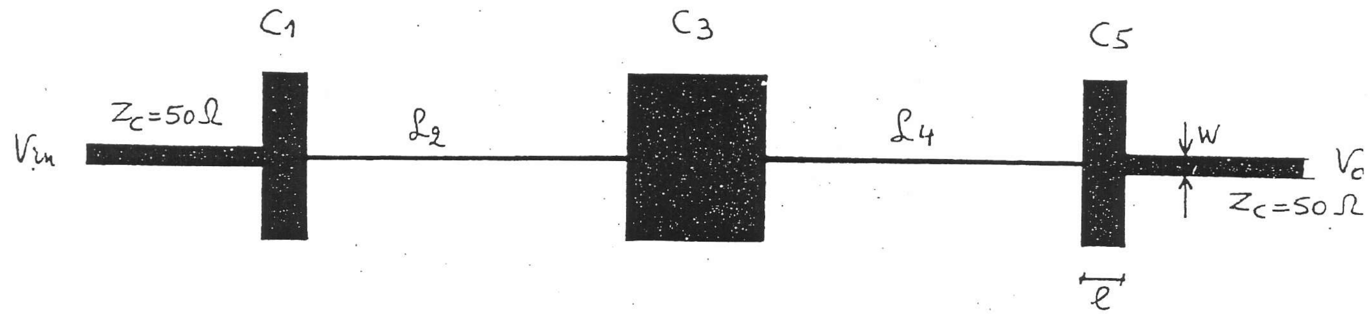


$C_1 = 3{,}94pF$; $\mathcal{L}_2 = 25{,}75nH$; $C_3 = 12{,}73pF$; $\mathcal{L}_4 = 25{,}75nH$; $C_5 = 3{,}94pF$

Fig.3-13: LPF with Butterworth, discreet components.

After some microstrip-calculations we get
inductances: chose Z_c=100Ω, (substrate epoxy ϵr=5, h=1.55mm) after some hammerstädt-calculation and Wheeler-graphs we have w/h=0.4, strip-length calculation gives l=42mm (value for $\mathfrak{L}$)
capacitors: chose Z_c=10Ω, we get for $\mathfrak{C}_{1,5}$ a w/h=15 and strip-length l=5,6mm; $\mathfrak{C}_3$ a w/h=15 and l=18mm.

For the 50Ω input-output-lines we get w/h=1.75.



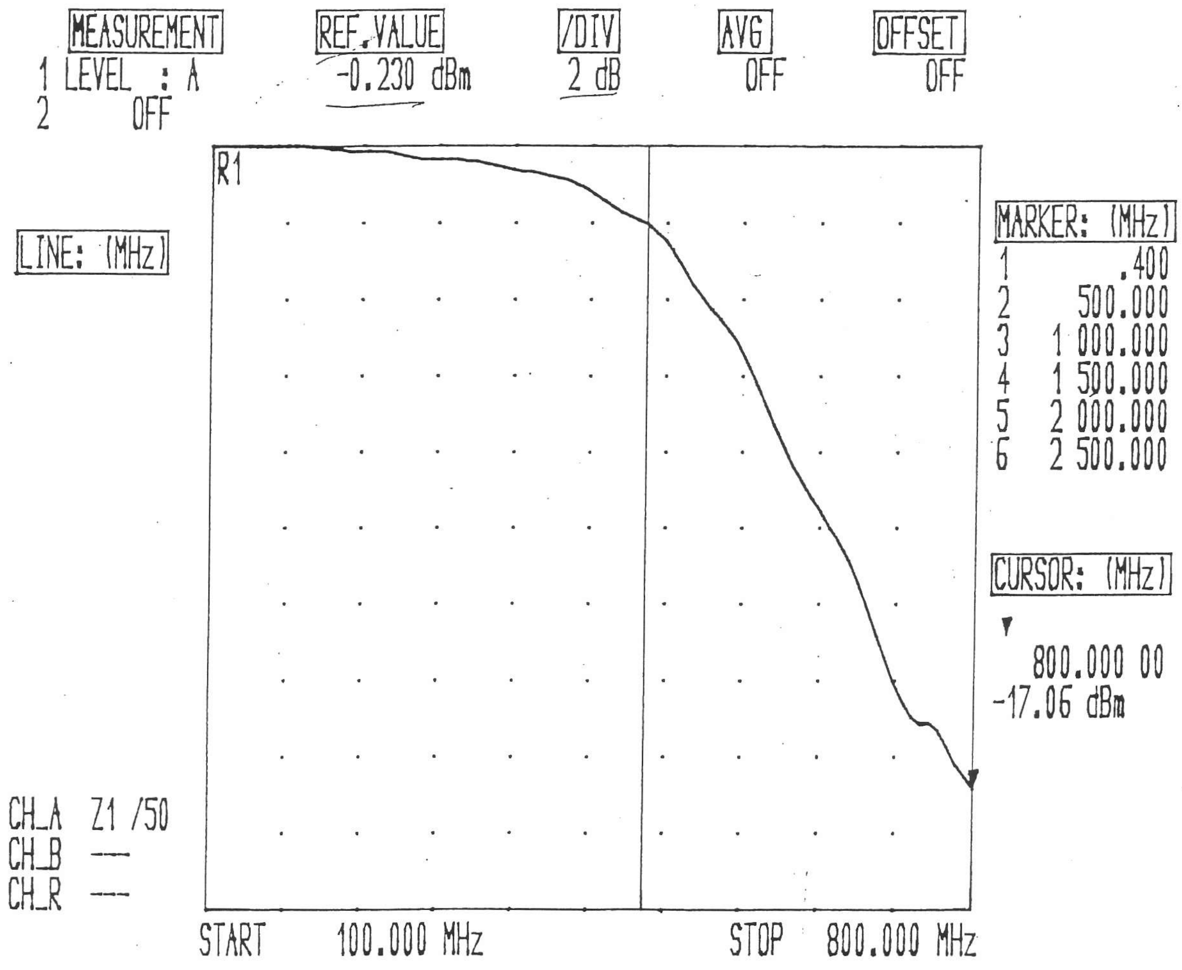


Fig.3-14: LPF in microstrip with Bode-plot.

3.2 Electromagnetic Waveguides

For low frequencies (max. 2 GHz) coax or microstrip satisfies as transmission line. For higher frequencies we use *wave-guides* (or radar-tubes). In this chapter we're only going to examine the rectangular waveguides.

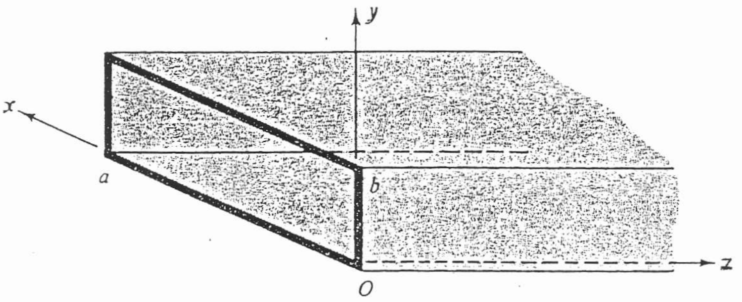


Fig.3-15: Rectangular waveguide.

The propagating waves are referred to as transverse electric and magnetic waves: TEM waves for short. We can distinguish between 3 components for the electric and magnetic fields: E_x, E_y, E_z; H_x, H_y, H_z. Now there are 3 different modes possible, TEM, TE, TM. In the TEM mode E and H are perpendicular to the propagation-direction z, so there are only the E_x, E_y, H_x, H_y components (so **no** E and H in the propagation-direction). In the TE mode only the E-field is perpendicular to the z-direction, so we have E_x, E_y, H_x, H_y, H_z. And for TM we have H_x, H_y, E_x, E_y, E_z existing.

It will be shown that TE and TM waves can exist inside hollow conducting tubes, and so are usually associated with waveguides.

3.2.1 Dimensions of waveguides

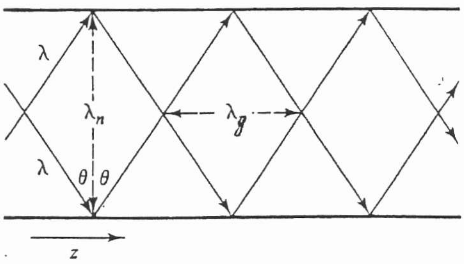


Fig.3-17: Wave propagating in guide.

$$v_n = c.\cos\theta$$
$$v_g = c.\sin\theta\ , \quad so\ v_g<c$$
$$\lambda_n.\cos\theta = \lambda \quad \rightarrow \quad \lambda_n = \frac{\lambda}{\cos\theta}$$
$$\lambda_g.\sin\theta = \lambda \quad \rightarrow \quad \lambda_g = \frac{\lambda}{\sin\theta} \qquad (3.12)$$

On which distance do we have to place the second plate with respect to the wavelength λ_n? There has to be one or an integer number of half-wavelengths between the two plates, as indicated in fig.3-18.
Let a be the distance between the plates, then

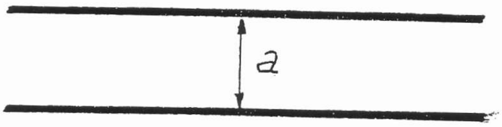


Fig.3-18: Distance a between the 2 plates of a waveguide.

$$a = m.\frac{\lambda_n}{2}$$
after some re-arangements we get
$$\rightarrow \lambda_g = \frac{\lambda}{\sqrt{1-(\frac{m\lambda}{2a})^2}}$$

So not all signals will propagate because the denominator can be zero (=no propagating wave in the direction of the guide).

$$\lambda_g = \frac{\lambda}{\sqrt{1-(\frac{\lambda}{\lambda_c})^2}} \quad with\ \lambda_c = \frac{2a}{m} \qquad (3.13)$$

so λ_c is a *critical wavelength*, ***the waveguide acts as a high-pass filter.***

- *example:* f=6 GHz, a=3 cm, dominant mode m=1

  There will be propagation if $\lambda<\lambda_c$, or $f>f_c$, with λ_c=2a/m=6 cm, and $\lambda=c/f=3.10^8/6.10^9$=5 cm.
  So a signal of 6 GHz will propagate through the wave-guide.

Fig.3-20: Modes m=1 and m=2.

Suppose m=2, now λ_c=3 cm and λ=5 cm, no propagation of a 6 GHz-signal occurs.

Now we're going to expand formule (3.13) for a distance *a* and *b* in a two-dimensional structure, shown in fig.3-21.

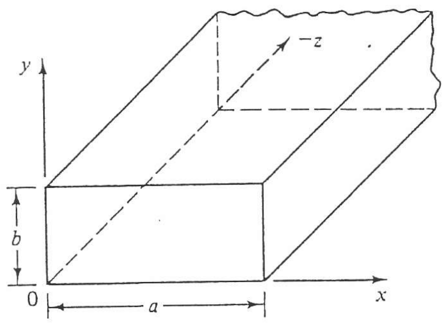


Fig.3-21: Rectangular waveguide with length and width a,b.

We get

$$\lambda_c = \frac{2}{\sqrt{(\frac{m}{a})^2 + (\frac{n}{b})^2}} = \frac{2\pi}{v_f} . f_c = \frac{\omega}{v_f} \quad with\ v_f = \frac{1}{\sqrt{\mu . \epsilon}}$$

$$\Rightarrow \omega_c = \frac{1}{\sqrt{\mu . \epsilon}} . \sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2} \qquad (3.14)$$

with propagation-condition $\omega > \omega_c$. Coefficients *m,n* indicate the numbers of half-wavelengths with respect to walls *a* and *b*.
Very important is to know the caracteristic impedance of a wave guide, knowing the *free-space impedance* = *377Ω*.

$$1)\ Z_c = \frac{Z_o}{\sqrt{1 - (\frac{\lambda}{\lambda_c})^2}} \qquad TE\text{-}MODUS \rightarrow Z_c \gg 377\,\Omega \quad (3.15)$$

$$2)\ Z_c = Z_o . \sqrt{1 - (\frac{\lambda}{\lambda_c})^2} \qquad TM\text{-}MODUS \rightarrow Z_c \ll 377\,\Omega \quad (3.16)$$

3.2.2 Wave-patterns in wave-guides

Helmholtz-equations: consider the rotor-Maxwell-equations in the frequency-range

$$\overline{\nabla}\wedge\overline{E} = -j\omega\mu\overline{H}$$

$$\overline{\nabla}\wedge\overline{H} = j\omega\epsilon\overline{E} \qquad (3.17)$$

$$\textit{we have } \overline{\nabla}\wedge(\overline{\nabla}\wedge\overline{E}) = -j\omega\mu(\overline{\nabla}\wedge\overline{H}) = +\omega^2\mu\epsilon\overline{E} \qquad (3.18)$$

This reduces to

$$\nabla^2\overline{E}+\omega^2\mu\epsilon\overline{E} = 0 \vee \nabla^2\overline{E}+k^2\overline{E} = 0$$

$$\textit{and } \nabla^2\overline{H}+k^2\overline{H} = 0$$

$$\textit{with } \nabla^2 = \textit{Laplacian} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \overline{\nabla}.\overline{\nabla}$$

2 Modes: $TE_{mode} \rightarrow$ $E_x.i_x+E_y.i_y$ (standing wave in the section)
$TM_{mode} \rightarrow$ $H_x.i_x+H_y.i_y$ (" ")

• *example:* TE-modus

$$\overline{E} = \underbrace{(E_x(x,y).\overline{i_x}+E_y(x,y).\overline{i_y})}_{\text{standing-wave pattern (x-y) section}}.\underbrace{\phi(z)}_{\text{propagation-condition eq.}} \qquad (3.19)$$

division of eq(3.19) by $E_x(x,y).\varphi(z)$ [for ease we write $E_x(x,y)=E_x$], gives

$$\nabla^2 E_x(x,y).\phi+k^2.E_x(x,y).\phi = 0$$

(eq. for x-direction)

$$\left(\frac{\partial^2.E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}\right).\phi(z)+E_x(x,y).\frac{\partial^2\phi(z)}{\partial z^2}+k^2.E_x.\phi(z) = 0$$

$$\frac{1}{E_x}.\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}\right)+\frac{1}{\phi(z)}.\frac{\partial^2\phi(z)}{\partial z^2}+k^2 = 0$$

$$\underbrace{\left(\frac{\partial^2 E_x}{\partial x^2}+\frac{\partial^2 E_x}{\partial y^2}\right)+A.E_x = 0}_{\text{I}} \qquad \underbrace{\frac{\partial^2\phi(z)}{\partial z^2}+(k^2-A).\phi(z) = 0}_{\text{II}}$$

(I): distribution of the field in the (x,y)-plane
(II): distribution along the z-axis

Solution of (I): **condition: $E_x(x,y) = \sin(m'x).\cos(n'y) = 0$**

So, if $x=0 \rightarrow \sin 0=0$
$x=a \rightarrow$ the conditions for $E_x=0$ are:
$m'.a=m.\pi \rightarrow m'=(m\pi)/a$
if $y=0 \rightarrow \sin 0=0$
$y=b \rightarrow$ the conditions for $E_y=0$ are:
$n'.b=n.\pi \rightarrow n'=(n\pi)/b$

$$\bar{E}_{x,y} = C_1.\sin(\frac{m\pi x}{a}).\cos(\frac{m\pi y}{b}).\bar{i}_x + C_2.\cos(\frac{m\pi x}{a}).\sin(\frac{n\pi y}{b}).\bar{i}_y \quad (3.20)$$

If we fill in the solution of the equation in (I), we find

$$\mathbf{A} = \mathbf{m'^2+n'^2} = \mathbf{k_c^2} \qquad (3.20) \qquad \text{(ALWAYS positive)}$$

with k_c^2 ($=2\pi/\lambda$) the critical wave-number.

By giving some specific values to *m* and *n* we can create different modes.

Solution of (II):

$$\phi(z) = A'.e^{-j.\sqrt{k^2-k_c^2}.z} + B'.e^{+j.\sqrt{k^2-k_c^2}.z}$$

$k^2>k_c^2$ the $\sqrt{}$ gets positive and there will be propagation,
$k^2<k_c^2$ the $\sqrt{}$ is negative and there will not be propagation.

$$k_c = \sqrt{(\frac{m\pi}{a})^2 + (\frac{m\pi}{b})^2}$$

$$= \frac{2\pi}{v}.f_c \quad so \quad f>f_c \quad (3.21)$$

3.3.3 The dominant mode

If we assume that *a* and *b* are unequal, and in particular *a>b*, form.(3.21) shows that the lowest cut-off frequency corresponds to the case m=1, n=0, when

$$f_{10} = \frac{v}{2a} \qquad (3.22)$$

The next lowest frequency will be either f_{20} or f_{11} depending on the relative magnitudes of *a* and *b*. There is thus a range of frequencies for which only one mode can propagate. This is called the *dominant mode*. It is instructive to express the cutt-off condition in terms of wavelength. Since v is the velocity of a TEM wave in the medium, the wavelength of a TEM wave of frequency f_{10} is given by $\lambda=v/f_{10}$.
Eq.(3.22) then shows that

$$\lambda = 2a$$

The cut-off frequency corresponds with the wavelength for which the width of the waveguide is one half-wavelength.
When a source of any sort is used to generate waves in a waveguide it will in general produce all modes in varying proportions.

However it is desired that only propagation in the lowest mode is possible. In cases where the frequency is such that propagation is possible at several modes, special precautions are needed, since departures from ideal guides will lead to one mode generating another, and *mode coupling* takes place.

For frequencies less than f_{mn} the fields are attenuated, with attenuation coefficient α nepers per metre given by

$$\alpha = \frac{2\pi}{v} \cdot \sqrt{f_{mn}^2 - f^2} \qquad (3.23)$$

3.3.4 TE modes

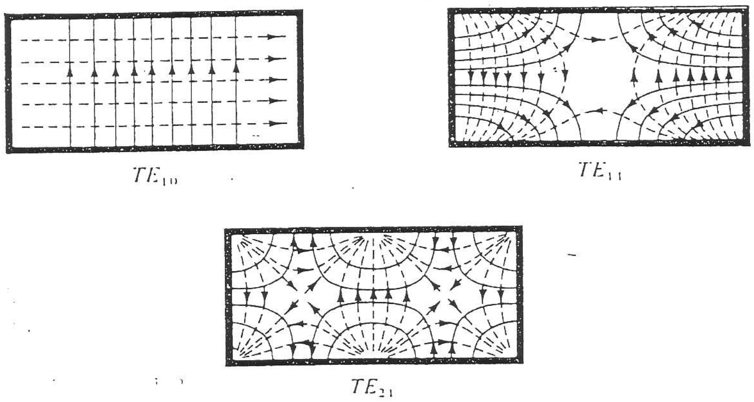


Fig.3-22: Transverse field patterns of TE modes in rectangular waveguide. Solid lines shows E-field, pecked lines H-fields.

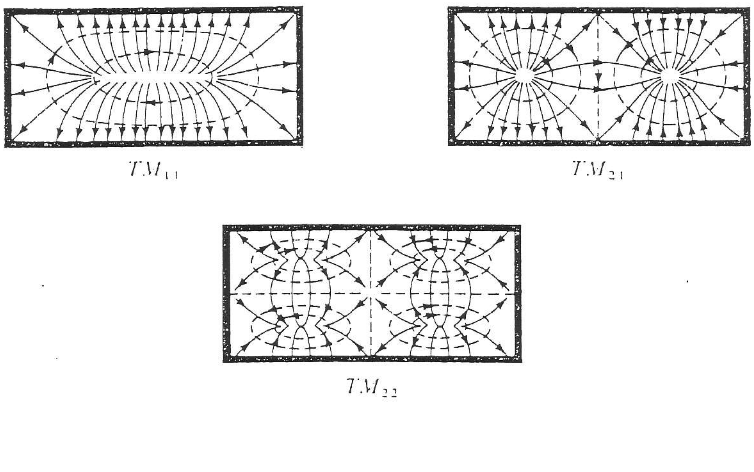


Fig.3-23: Transverse field patterns of TM modes in rectangular waveguide.

• *example:*

Some standard rectangular guides

Type	Inside dimensions (in)	(mm)	TE_{10} cut-off (GHz)	Range (GHz)
WG 10	2.840 × 1.340	72.14 × 34.04	2.080	2.60–3.95
WG 12	1.872 × 0.872	47.55 × 22.15	3.155	3.95–5.85
WG 14	1.372 × 0.622	34.85 × 15.80	4.285	5.85–8.20
WG 16	0.900 × 0.400	22.86 × 10.16	6.56	8.20–12.4
WG 18	0·622 × 0.311	15.80 × 7.90	9.49	12.4–18.0
WG 22	0.280 × 0.140	7.11 × 3.56	21.10	26.5–40.0

WG 16 has internal dimensions 22.9 mm × 10.2 mm and is air-filled. Find the five lowest cut-off frequencies. It is recommended for use in the dominant mode for frequencies between 8.20 and 12.40 GHz. Find the phase velocity and guide wavelength at these extreme frequencies in terms of free-space *TEM* values.

$$a = 0.0229 \text{ m}$$
$$b = 0.0102 \text{ m}$$
$$v = c = 3 \times 10^8 \text{ m s}^{-1}$$

We find

$$f_{10} = 6.55 \text{ GHz}$$
$$f_{20} = 13.1 \text{ GHz}$$
$$f_{01} = 14.7 \text{ GHz}$$
$$f_{11} = 16.0 \text{ GHz}$$
$$f_{21} = 19.7 \text{ GHz}$$

$$v_{10}/c = f/(f^2 - f_{10}^2)^{\frac{1}{2}}$$

We have

$$\lambda_g = v_{10}/f = c/(f^2 - f_{10}^2)^{\frac{1}{2}}$$

Hence

$$\lambda_g/\lambda = f/(f^2 - f_{10}^2)^{\frac{1}{2}}$$

At 12.40 GHz

$$v_{10}/c = \lambda_g/\lambda = 1.178$$

At 8.20 GHz

$$v_{10}/c = \lambda_g/\lambda = 1.662$$

3.3.5 Resonant cavity

We consider a length *d* of rectangular waveguide of cross-section a×*b* which is closed at the two ends by conducting plates, as shown in fig.3-24.

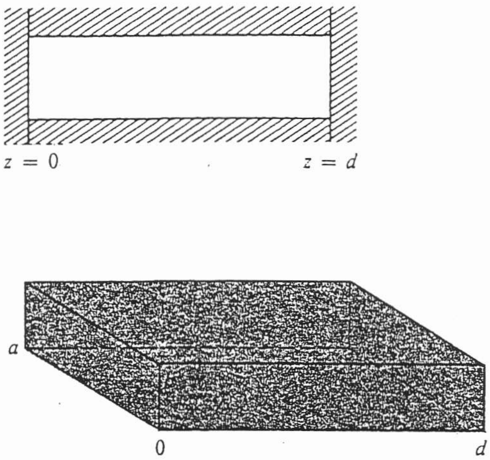


Fig.3-24: A resonator in rectangular waveguide.

Hence

$$\omega_c = \frac{\pi}{\sqrt{\mu . \epsilon}} \cdot \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{l}{d}\right)^2} \qquad (3.24)$$

The 'box' resonates at discrete frequencies, and not in a continuous range as a $\mathcal{L}\mathcal{C}$-resonator circuit.
With a real cavity, losses occur in the walls and also in the connection used to couple the cavity to source and load. These losses produce a range about each resonant frequency over which the cavity can be excited, as is true for resonant circuits.

3.3.5 Waveguide systems and circuits

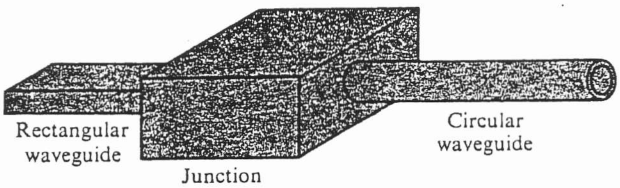


Fig.3-25: Junction between two different waveguides.

It is desirable *to match waveguides* in the same way as it is desirable *to match transmission lines;* for example, at high power the VSWR should be near unity to avoid excessive locl stresses, or a receiving aerial should be matched to the waveguide feed to abtain maximum received signal. As with transmission line, we need to introduce admittances in shunt at points along the waveguide. Such admittances can conviently be produced by posts and by thin diaphragms. Some examples are shown on the next page.

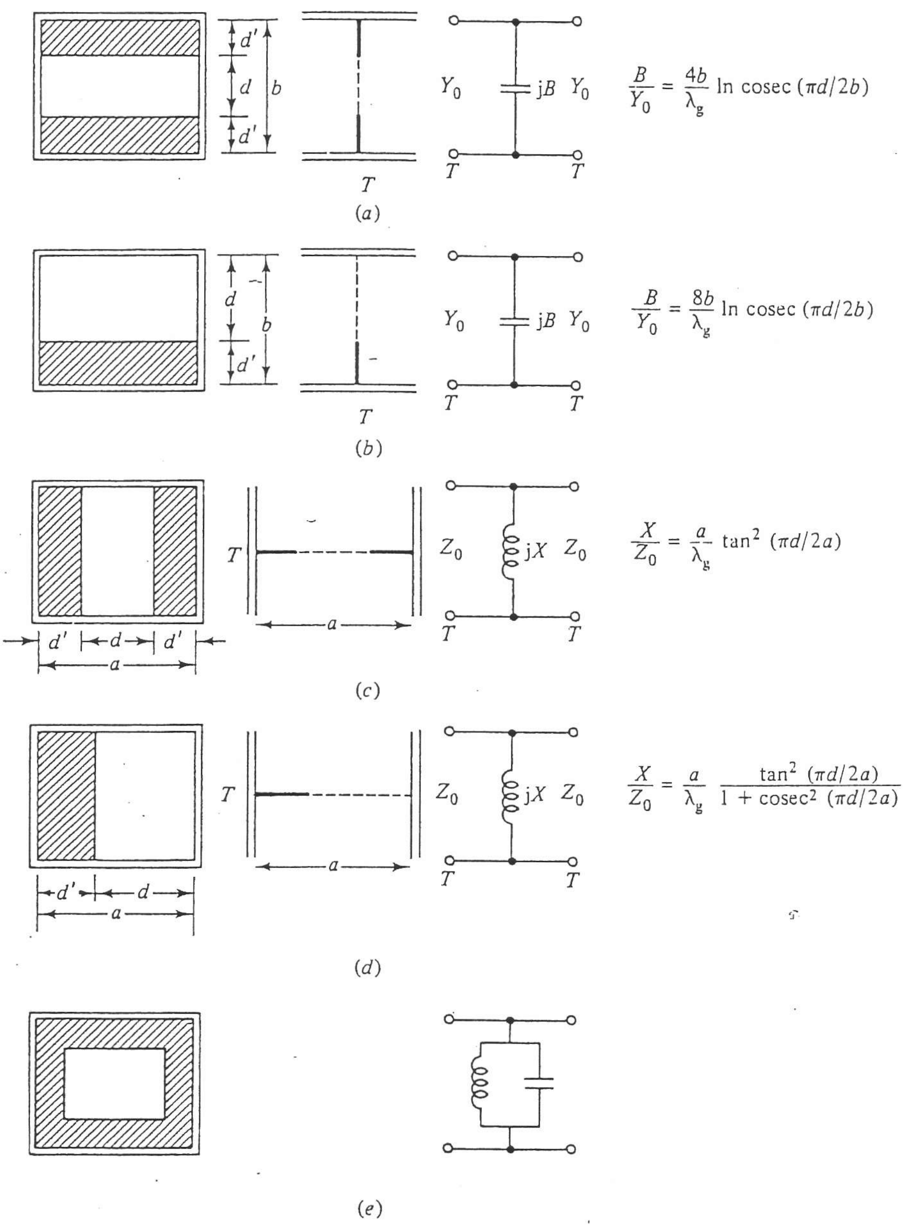


Fig.3-26: Irises in waveguide as admittances. (a)Symmetrical capacitive.(b)Asymmetrical capacitive.(c)Symmetrical inductive.(d) Asymmetrical inductive.(e)Resonant.

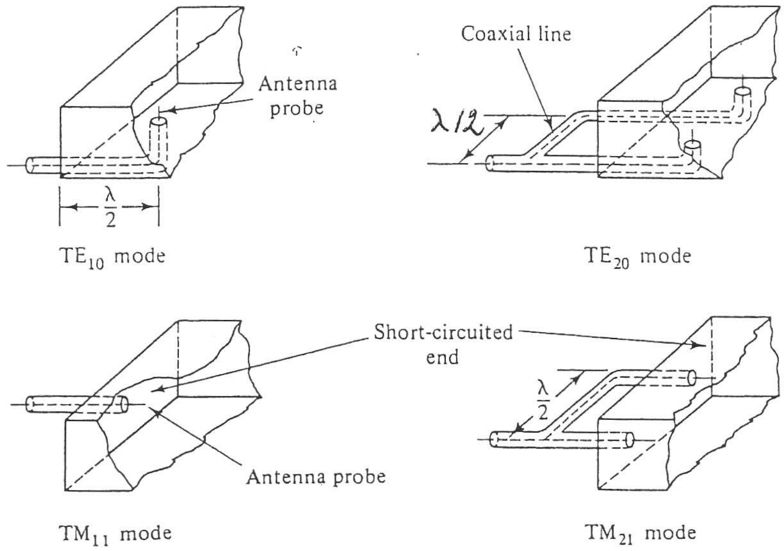


Fig.3-27: Methods of exciting various modes in rectangular waveguides.

A reflectionless termination is made by arranging for the gradual absorption of the incident wave. A resistive sheet forms a gradual taper in a plane containing the electric field.

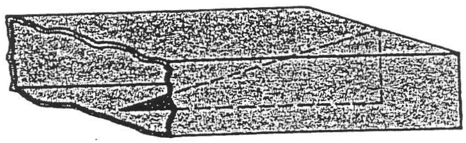


Fig.3-28: Reflectionless load for rectangular waveguide.

It is often necessary to couple small amounts of energy between two waveguides or parts of the same guide. Sometimes the coupling may be into free space. Such coupling may be done by small holes or slots in correctly chosen parts of the waveguide wall, or by placing an iris with a small window across the guide.

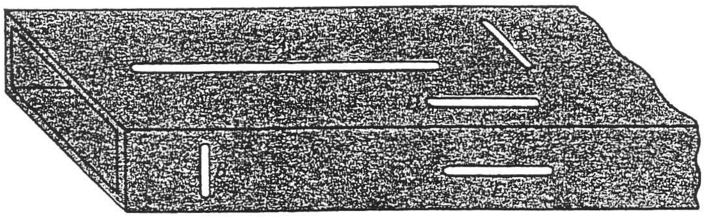


Fig.3-29: Disposition of slots.

Connections are useful in many ways: as a means of adding components for matching purposes or for power sharing between several loads.

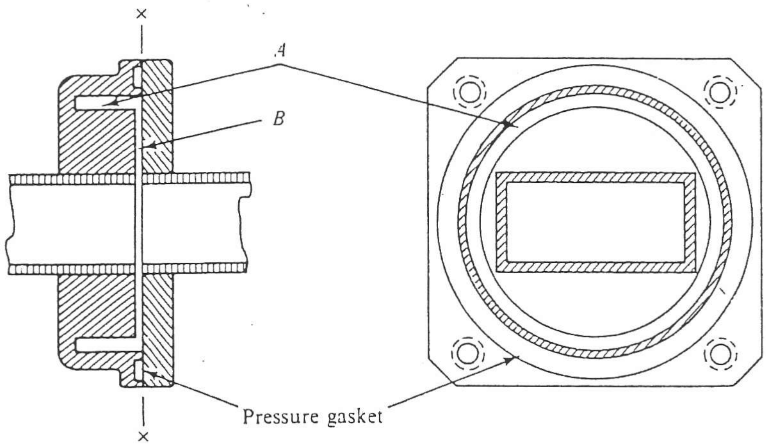


Fig.3-30: Choke coupler.

A magic tee (Fig.3-31) is commonly used for mixing, duplexing, and impedance measurements. A magic tee may be used to couple the two transmitters to the antenna in such a way that the transmitters do not load each other. The two transmitters should be connected to ports 3 and 4, respectively. Transmitter 1, connected to port 3, causes a wave to emanate from port 1 and another to emanate from port 2; these waves are equal in magnitude but opposite in phase. Similary, transmitter 2, connected to port 4, gives rise to a wave at port 1 and another at port 2, both equal in magnitude and in phase. At port 1 the two opposite waves cancel each other. At port 2 the two in-phase waves add together; so double output power at port 2 is obtained for the antenna.

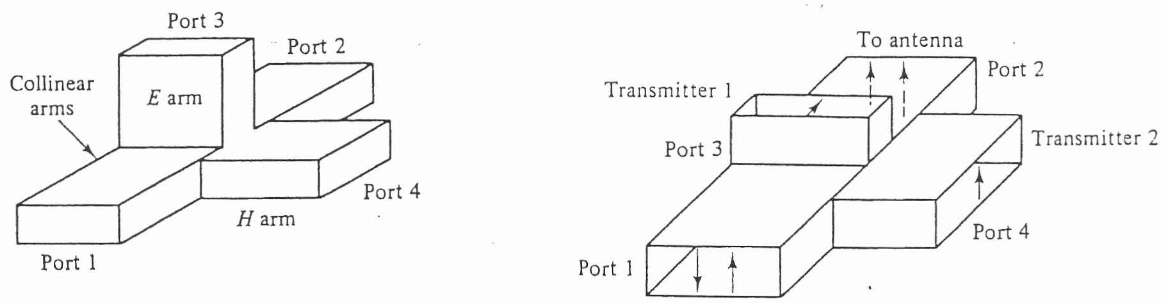


Fig.3-31: The magic-T.

Another junction is the hybrid-ring. If it is remebered that one half-wavelength of line has the effect of changing the sign of voltage and current between input and output, it will be seen that the four ports are symmetrical save for a phase inversion between 1 and 4.
Feeding from port 1 will therefore cause in a load on port 3 currents through the two paths which are equal and opposite, so that no coupling takes place to port 3. Using the impedance-inverting property of a quarter-wavelength line, we can see that opposite ports are not coupled.

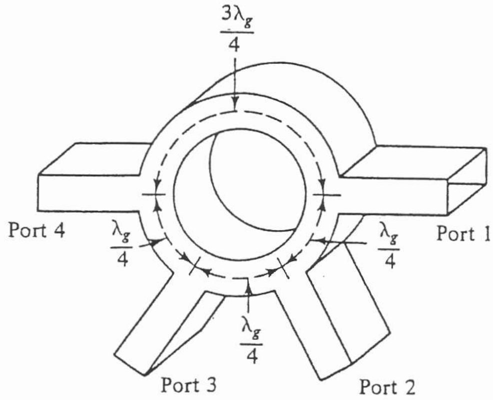
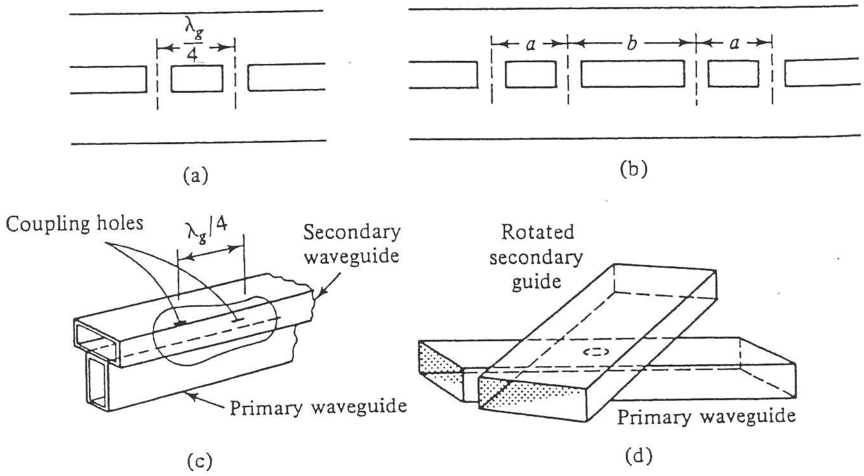
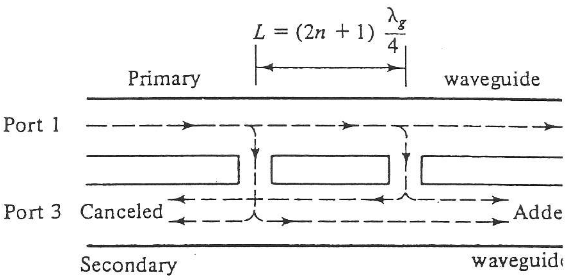


Fig.3-32: The hybrid-ring.

Some other possiblities :



Different directional couplers. (a) Two-hole directional coupler. (b) Four-hole directional coupler. (c) Schwinger coupler. (d) Bethe-hole directional coupler.



Two-hole directional coupler.

3.3 Optical Waveguides (fiber-optics)

An optical fiber transmission link comprises the elements shown in fig.3-33. The key sections are a transmitter consisting of a light source and its associated drive circuitery, a cable offering mechanical and environmental protection to the optical fibers contained inside, and a receiver consisting of a photodetector plus amplification and signal-restoring circuitry. A practical optical fiber generally contains several cylindrical hair-thin glass fibers, each of which is an independent communication channel.

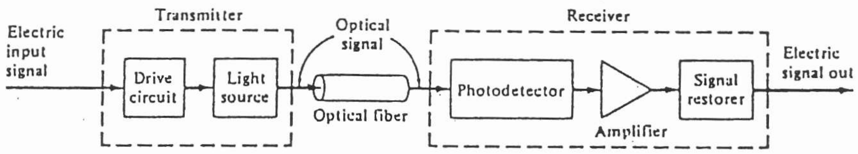


Fig.3-33: Basic elements of an optical fiber transmission link.

A fundamental optical parameter of a material is the *refractive index* (or *index of refraction*). In free space a light wave travels at a speed $c=3.10^8$m/s. The speed of light is related to the frequency f and the wavelength λ by $c=f\lambda$. The ratio of the speed of light in a vacuum to that in matter is the index of refraction $\mathbf{n}$ of the material and is given by

$$n = \frac{c}{f} \qquad (3.25)$$

Typical values are n=1 for air, 1.33 for water, 1.5 glass, 2.42 diamond.
The relationship at the interface is know as Snell's law and is given by

$$n_1 sin\varphi_1 = n_2 sin\varphi_2 \qquad (3.26)$$

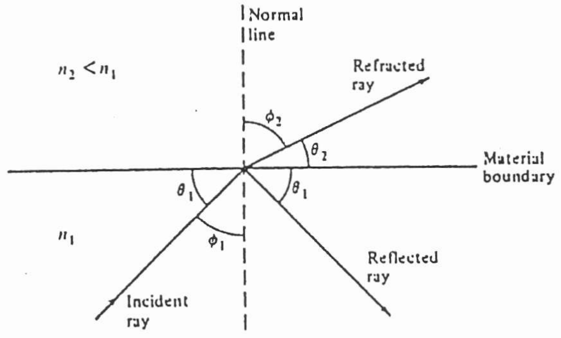


Fig.3-34: Refraction and reflection of a light ray at material boudary.

As the angle of incidence φ_1 in an optically denser material (higher refractive index) becomes smaller, the refracted angle φ_2 approaches zero. Beyond this point no refraction is possible and the light rays become *totally internally reflected.*

This point is known as the critical angle of incidence φ_c.

$$\varphi_c = \arc \cos \frac{n_2}{n_1} \qquad (3.27)$$

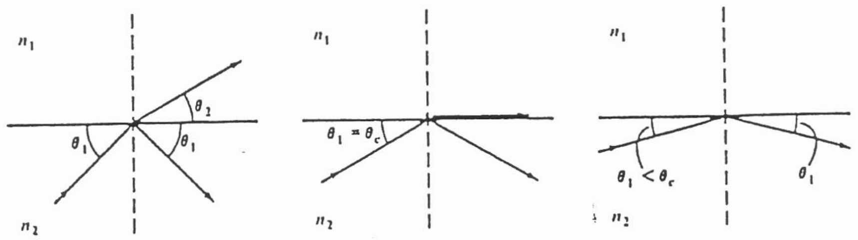


Fig.3-35: Representation of the critical angle and total internal reflection at a glass-air interface.

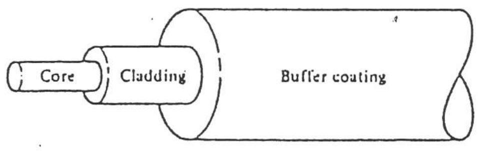


Fig.3-36: Schematic of a single-fiber structure. A circular solid core of refractive index n_1 is surrounded by a cladding having a refractive index $n_2<n_1$.

Variation in the material composition of the core give rise to the two commonly used fiber types shown in fig.3-37. In the first case the refractive index of the core is uniform throughout and undergoes an abrupt change (or step) at the cladding boundary. This is called a *step-index fiber*. In the second case the core refractive index is made to vary as a function of the radial distance from the center of the fiber. This type is a *graded-index fiber*.

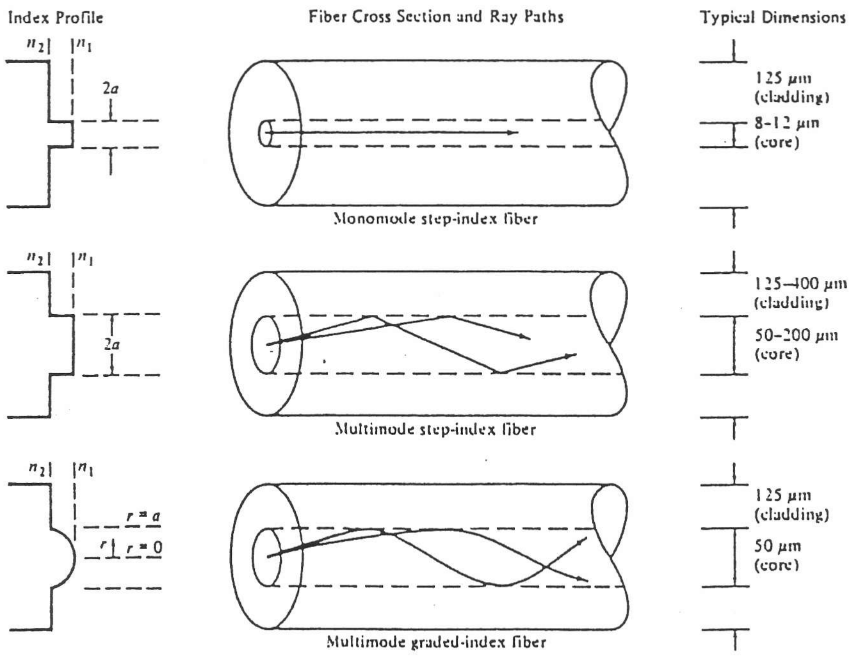


Fig.3-37: Comparison of single-mode and multimode step-index and graded-index optical fibers.

The step-index fibers can be further divided into singlemode and multimode classes. As the name implies, a single-mode fiber sustains only one mode of propagation, whereas multimode fibers contain many hundreds of modes. The advantages of multimode are

- easier to launch optical power into the fiber
- so easier to connect similar fibers
- use of LED, whereas single-mode requires laser diodes.

The disadvantage compared to single-mode fibers is the *intermodal dispersion*. When an optical pulse is launched into the fiber, the optical power in the pulse is distributed over all of the modes, (so each over a slightly different velocity) thus causing the pulse to spread out in time at the output.

From Snell's law the minimum angle φ_{min} that supports total internal reflection for the meridional ray is

$$\sin(\varphi_{min}) = \frac{n_2}{n_1}$$

$$n.\sin\theta_{o,max} = n_1 \sin\theta_c = (n_1^2 - n_2^2)^{\frac{1}{2}}$$

$$NA = n.\sin\theta_{o,max} = (n_1^2 - n_2^2)^{\frac{1}{2}} \approx n_1\sqrt{2\Delta} \qquad (3.28)$$

Equation (3.28) defines the *numerical aperture* NA of a step-index fiber for meridional rays.

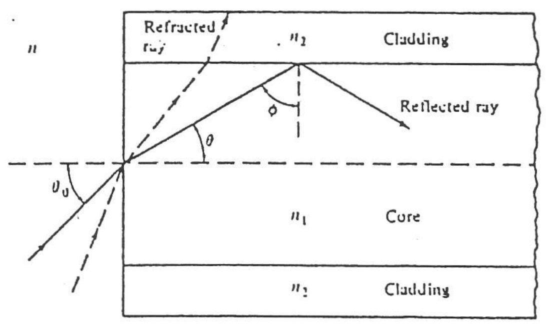


Fig.3-38: Meridional ray optics representation of the propagation mechanism in an ideal step-index fiber.

When the NA-value is high, the fiber collects more radiation from the light-source.